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THE GEOMETRY OF THE WEDDLE MANIFOLD W_p*

BY A. B. COBLE

During the last few years our secretaries have been so active in organizing interesting programs at the regular meetings that surveys of progress in the more important fields of mathematics have been presented to the Society quite recently. At the meeting last April in New York, reports on Algebraic Geometry were given. I have therefore felt impelled to change in some measure the traditional character of this biennial address, and to select a topic somewhat more special than usual. Lest this justification may seem too hollow, let me add that this change is in the direction of my personal interest in mathematics. For the benefit of those members present who are not specialists in geometry, I hope to illustrate by relatively simple early cases the general aspect of the problem, emphasizing those methods whose extension presents no difficulty. A technical account of the novelties presented will appear in the April number of the American Journal of Mathematics.

The Weddle manifold W_p has the dimension p in the linear space S_{2p-1}, the first case for $p=2$ being the long known Weddle quartic surface in S_3. Let us consider first its group-theoretic character. We all are familiar with the quadratic Cremona involution,

$$I_{yz}: \qquad x_i x_i' = y_i z_i, \qquad (i = 0, 1, 2; x, x' \text{ in the same plane}),$$

which has F-points at the vertices of the triangle of reference and which interchanges the points y and z. Its fundamental importance is due to the fact that the entire group of Cremona transformations in the plane is generated by the group of collineations and by the single element I_{yz} above. For every larger limit for i, $(i=0, 1, 2, \cdots, r)$, this involution I_{yz} persists; and it, together with the group of collineations, generates the *regular* Cremona group in S_r. When $r \geqq 3$, this regular Cremona group is no longer the entire Cremona group of the space. The regular

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Cremona transformations are characterized by the property that they are completely defined to within collineations at most by a finite number of F-points, the existence of other F-loci, F-curves, F-surfaces, etc., being a necessary consequence of the existence of these F-points.

Returning to S_2 we take $2+3$ points, $p_1, p_2, \cdots, p_5 = P_6^2$, and denote by I_{12} that involution I_{yz} which interchanges p_1 and p_2 and which has F-points at p_3, p_4, p_5. Then the ten involutions I_{ij}, $(i, j = 1, \cdots, 5;\ i \neq j)$, generate a finite abelian group of order 2^4 and type $(1, 1, 1, 1)$ whose elements multiply according to the following rules:

$$I_{ij}^2 = 1,\ I_{ij}I_{ik} = I_{jk},\ I_{ij}I_{kl} = I_{kl}I_{ij} = I_{ik}I_{jl} = I_{ijkl}.$$

It consists of the identity, the ten quadratic involutions, and five cubic involutions of type I_{ijkl} each with one double and five simple F-points. Similarly in S_3, with $3+3$ points, $p_1, \cdots, p_6 = P_6^3$, let I_{12} be that cubic involution I_{yz} which interchanges p_1 and p_2 and which has F-points at $p_3, \cdots, p_6$. We again obtain a finite abelian group of order 2^5 with a multiplication table as above. Proceeding in this fashion we define in terms of $r+3$ points in S_r an abelian group of order 2^{r+2}.

There is however an interesting difference between the spaces of odd and of even dimension. When r is odd, say $r = 2p-1$, the number of points being $2p+2$, the abelian group contains a symmetric element $I_{12\cdots 2p+2}$ for which the F-points all have the same character. These symmetric regular Cremona transformations are very scarce. In every S_r, $(r \geqq 2)$, there is the generating element I_{yz}. In every odd S_{2p-1} there is the element $I_{12\cdots 2p+2}$ just mentioned. Apart from these fairly generic types there are only three further symmetric types in S_2, one further type in S_3, and one in S_4. We define the procession of Weddle manifolds W_p, $(p \geqq 2)$, to be *the loci of fixed points of the involutions* $I_{12\cdots 2p+2}$. Thus the first Weddle W_2 in S_3 is the locus of fixed points of that space involution whose pairs are the seventh and eighth base points of nets of quadrics contained in the web on P_6^3, these two points coming together on the locus of nodes of quadrics on P_6^3.

Because of the abelian character of the finite Cremona G_2^{2p+1}, W_p is invariant under the group, and on W_p each point is invariant under the identity and $I_{1\cdots 2p+2}$. Thus G_2^{2p+1} effects

upon W_p the transformations of a g_2^{2p} which is isomorphic with the additive group of half periods of the theta functions of genus p, so that some relation of W_p to these functions may be expected. Since also the set of points P_{2p+2}^{2p-1}, which defines $I_{12\cdots 2p+2}$ and therefore W_p, is on a unique rational norm-curve N^{2p-1}, and may be projectively determined on this curve by a binary $(2p+2)$-ic, $(\omega t)^{2p+2}=0$, it is clear that the theta functions involved would necessarily be the *hyperelliptic thetas*.

Let us turn then to the hyperelliptic curves for a second definition of W_p. The normal form commonly used for the discussion of the integrals on such a curve is

$$z^2 = (\omega t)^{2p+2} = (t\,t_1)\cdots(t\,t_{2p+2}),$$

the unique g_1^2 on the curve being given by z, $-z$. But another normal form has also been much used, namely, that for which the unique g_1^2 is cut out by a pencil of lines, that is

$$H_p^{p+2} \equiv f_p(t_1, t_2)\cdot y_0^2 + 2f_{p+1}(t_1, t_2)y_0 + f_{p+2}(t_1, t_2) = 0,$$

f_p, f_{p+1}, f_{p+2} being binary forms in t_1, t_2, of the orders indicated. This curve of genus p and order $p+2$ has a p-fold point at $O(y_0:t_1:t_2=1:0:0)$. The branch lines of the g_1^2 are given by

$$(\omega t)^{2p+2} = f_{p+1}^2 - f_p f_{p+2} = \prod(tt_i) = 0,$$

and the $2p+2$ branch points on the curve by

$$y_0:t_1:t_2 = -f_{p+1}(t_i):t_{i1}f_p(t_i):t_{i2}f_p(t_i).$$

The $3p+5$ constants in the forms f are reduced to $p+3$ if the branch lines, $(\omega t)^{2p+2}=0$, are given. Three more constants can be removed by a projectivity which leaves fixed the p-fold point O and the branch lines on it. Under these conditions H_p^{p+2} has p absolute projective constants. The ∞^p curves obtained by variation of these absolute projective constants are all birationally equivalent. It may be shown that every member of the aggregate (∞^p) may be obtained from a given one by a Cremona transformation and that such member is uniquely determined when a given p-ad of points on the given curve is required to pass into the p-ad at O. If the branch points of H_p^{p+2} are $r_1, \cdots, r_{2p+2}$, then the set of $2p+3$ points in S_2 consisting of these $2p+2$ branch points and O are *associated* with a set of

$2p+3$ points in S_{2p-1} of which $2p+2$ are the set P_{2p+2}^{2p-1} which defines W_p and the last is a point x on W_p which runs over W_p as the p-ad on the given $H_p{}^{p+2}$ varies.

I should like to comment briefly on this matter of the association of two sets of n points. In one of the simplest problems of algebra, namely, the solution of a system of linear homogeneous equations in n variables, we naturally drop the dependent equations, and we seek the solutions of the system of r linearly independent remaining equations. These depend upon the matrix, $M_{r,n}=(a_{ij})$, of coefficients of the system whose rank is r, say $r<n$ to avoid trivialities. The solutions of this system are given in terms of $n-r$ linearly independent solutions which may be arrayed in a matrix $M_{n-r,n}=(b_{ij})$. Obviously these two matrices are reciprocally related, the row product of any row of the one with any row of the other is zero. Thus the determination of the solutions (b_{ij}) of a system with coefficients (a_{ij}) carries with it the determination of the solutions (a_{ij}) of a system with coefficients (b_{ij}). This interesting reciprocity between the two matrices, which I shall call *apolar matrices*, passes practically unnoticed in treatises on algebra despite its important applications in geometry. One of these applications, given by Grassmann in 1862, is to the determination of a linear space on the one hand by means of the points which it contains, and on the other by means of the linear spaces which contain it. Another application is found in the extensive theory of apolarity of linear systems of algebraic forms and of linear systems of algebraic loci. In analysis we find that the matrix $M_{n-1,n}$, whose rows are 0th, 1st, $\cdots$, $(n-2)$th derivatives of the n solutions of a linear differential equation of the nth order, has, for its apolar matrix $M_{1,n}$, n solutions of the adjoint equation.

In the definition of apolar matrices the emphasis is on the rows. But, in a situation like this, a reflective geometer will not overlook the role played by the columns. The columns of $M_{r,n}$ may be interpreted as n points in S_{r-1}, those of $M_{n-r,n}$ as n points in S_{n-r-1}. These two sets of points, ordered with respect to each other, I have called *associated sets of n points*. Since in either matrix the rows can be replaced by independent linear combinations of all the rows, each of the two associated sets is determined by the other set only to within projectivities. The proportionality of complementary determinants of orders

r, $n-r$ in two apolar matrices shows that, if one set of points satisfies some geometric condition, the associated set must also satisfy some geometric condition, though this second condition is usually of a totally different geometric character from the first.

This association of two sets of n points has a deeper significance than its projective definition would indicate. The association is in fact invariant under properly applied regular Cremona transformation. If an involution I_{yz} with F-points at r of the first set transforms the remaining $n-r$ points into new points, these r F-points and $n-r$ new points form a new set of n points in S_{r-1} congruent to the original set. In the associated set in S_{n-r-1} take the $n-r$ complementary points as F-points of I_{yz} to form a similar congruent set. Then the two new sets, congruent respectively to the original associated sets, are still associated. Thus, in the two associated sets previously mentioned, first of P_{2p+2}^{2p-1} and x on W_p in S_{2p-1}, and second of $r_1, \cdots, r_{2p+2}$ and O in S_2, the involution I_{12} is associated with the perspective involution A_{12} with F-points at O, r_1, r_2 and with directions at r_1, r_2 corresponding respectively to points on the lines Or_1, Or_2. To the involution $I_{12\cdots 2p+2}$, of which x is a fixed point, there is associated the product $A_{12}\ A_{34}\cdots A_{2p+1,2p+2}=J_{12\cdots 2p+2}$, and this is the de Jonquiéres involution for which H_p^{p+2} is a locus of fixed points and for which the set of points r_i, O is congruent to itself.

We use the association, proved in the above manner, to obtain the coordinates of the point x on W_p. As a coordinate system in S_{2p-1} we take the $2p$ coefficients of a binary $(2p-1)$-ic, $(\alpha t)^{2p-1}$, the perfect powers of linear forms then representing the points of a rational norm-curve N^{2p-1}, which for present purposes we identify with the N^{2p-1} on the $2p+2$ points P_{2p+2}^{2p-1}. Coordinate systems of this character date back to Weyr and his contemporaries. They are especially effective for all problems in which a rational norm-curve enters in a significant way, since then the highly developed algebra of binary forms in one or more sets of variables can be utilized.

It is sufficient by way of illustration to take the case $p=2$. The $H_2^4=f_2y_0^2+2f_3y_0+f_4=0$ with double point O has for matrix of coordinates of its six branch points and of O:

$$\begin{pmatrix} -f_3(t_1) & -f_3(t_2) & \cdots & -f_3(t_6) & 1 \\ t_1 f_2(t_1) & t_2 f_2(t_2) & \cdots & t_6 f_2(t_6) & 0 \\ f_2(t_1) & f_2(t_2) & \cdots & f_2(t_6) & 0 \end{pmatrix},$$

t here being non-homogeneous for convenience. The matrix of coordinates of the six points, P_6^3, in S_3 on the norm-curve N^3 and of the point x on W_2 is the matrix of coefficients of the seven binary cubics

$$(tt_1)^3,\ (tt_2)^3,\ \cdots,\ (tt_6)^3,\ (\alpha t)^3.$$

We wish to determine $(\alpha t)^3$ and such constant factors for the perfect cubes that these two matrices may be apolar. Let the linear factors of $f_2(t)$ be

$$f_2(t) = (tr_1)\cdot(tr_2).$$

Then the linear identity connecting eight sixth powers reads:

$$\frac{(tt_1)^6}{f_2(t_1)\cdot\omega'(t_1)} + \cdots + \frac{(tt_6)^6}{f_2(t_6)\cdot\omega'(t_6)} + \frac{(tr_1)^6}{\omega(r_1)\cdot f_2'(r_1)} + \frac{(tr_2)^6}{\omega(r_2)\cdot f_2'(r_2)} \equiv 0,$$

where $f_j'(s)$ is the product of the differences of a particular root s of f_j and of each of the other roots of f_j. The polar of $f(t)$ as to this identity cancels the last two terms and removes the factor $1/f_2(t_i)$ from the others yielding the identity connecting six fourth powers. This identity shows that the six perfect cubes of the second matrix must have factors $1/f_2(t_1),\ \cdots,\ 1/f_2(t_6)$, respectively, in order that the second matrix may be apolar to the last two rows of the first matrix. That it may be apolar to the first row also requires that

$$(\alpha t)^3 = \frac{f_3(t_1)\cdot(tt_1)^3}{f_2(t_1)\cdot\omega'(t_1)} + \cdots + \frac{f_3(t_6)\cdot(tt_6)^3}{f_2(t_6)\cdot\omega'(t_6)}.$$

We have here then the coordinates of a point on the Weddle W_2 in S_3 in terms of the roots $t_1,\ \cdots,\ t_6$ of $(\omega t)^6 = f_3^2 - f_2 f_4$. It is not the best form of $(\alpha t)^3$, since there are many linear relations connecting six cubes, and since f_3 and f_2 satisfying

$$z_t^2 = (\omega t)^6 = f_3^2 - f_2 f_4$$

cannot be chosen arbitrarily when $(\omega t)^6$ is given. But, for a root r_i of f_2, $z_{r_i}=f_3(r_i)$. If then we polarize the above identity as to $f_3(t)$, the first six terms yield $(\alpha t)^3$, whence

$$-(\alpha t)^3 = \frac{(tr_1)^3}{z_{r_1}\cdot f_2'(r_1)} + \frac{(tr_2)^3}{z_{r_2}\cdot f_2'(r_2)}.$$

As immediate generalizations we find that

(A) $$(\alpha t)^{2p-1} = \sum_{i=1}^{i=2p+2} \frac{(tt_i)^{2p-1}\cdot f_{p+1}(t_i)}{f_p(t_i)\cdot \omega'(t_i)},$$

(B) $$-(\alpha t)^{2p-1} = \sum_{j=1}^{j=p} \frac{(tr_j)^{2p-1}}{z_{r_j}\cdot f_p'(r_j)}.$$

Since the coefficients α in this last formula are symmetric in the p pairs of values r_i, z_{r_i}, the coordinates x of a point on W_p are proportional to abelian functions of $u_1, \cdots, u_p$ on $H_p{}^{p+2}$ determined by the p-ad of points on $H_p{}^{p+2}$:

$$r_1, z_{r_1}; r_2, z_{r_2}; \cdots; r_p, z_{r_p}.$$

The *superposed* p-ad for which all the z's change sign yields the same point $(\alpha t)^{2p-1}$, whence the abelian functions are even.

The last formula expresses the coordinates of a point x on W_p in terms of the coordinates of the p crossings of N^{2p-1} and of the unique p-secant S_{p-1} of N^{2p-1} which passes through x. This S_{p-1} is determined by $f_p=0$. The S_{p-1} contains, however, a set of 2^{p-1} points x of W_p which arise from the variation of signs of z_{r_i}. We have then theorems of the following type:

The Weddle quartic $W_2{}^4$ in S_3 contains N^3 on $P_6{}^3$, and a bisecant of N^3 meets $W_2{}^4$ in two further points harmonic to the two crossings of the bisecant.

The Weddle W_3 in S_5 of order 19 *contains N^5 on $P_8{}^5$ as a triple curve. Of the* 19 *points in which it meets a trisecant plane of N^5,* 9 *are found at the three triple points on N^5, and* 6 *at the contacts with W_3 of the three bisecants at their crossings. The remaining* 4 *are generic points of W_3 whose diagonal triangle is the three crossings of the trisecant plane.*

Naturally this last theorem is in much more precise form than can be verified here.

Each element of a finite Cremona group in S_2 has a certain number of F-points and corresponding P-curves. The aggregate of such F-points and P-curves for all elements of the group must divide into one or more sets such that in a set all the members are conjugate under the group. In space S_3 we shall have F-points, F-curves, and P-surfaces similarly disposed under a finite group. We lump all of these under the generic term *F-loci of the group*. For regular groups in S_3, however, the F-points and P-surfaces must lie in a different conjugate system from the F-curves. Thus, for the $G_{2\cdot 2^4}$ of the Weddle W_2 defined by P_6^3 the directions at the point p_1, say π_1, pass by I_{56} into π_{156}, the plane on p_2, p_3, p_4; and this plane passes by I_{34} into π_{123456}, the quadric with node at p_2 and on the other five points. These 6 points, 20 planes, and 6 nodal quadrics make up a conjugate set of $2\cdot 2^4$ F-loci *of the first kind* conjugate under $G_{2\cdot 2^4}$. Similarly the line p_1p_2, say $\pi_{12}{}^{(2)}=\pi_{3456}{}^{(2)}$, passes by I_{12} into $\pi^{(2)}=\pi_{123456}{}^{(2)}$, the cubic curve N^3 on P_6^3. Again these 15 lines and one cubic curve make up a conjugate set of 2^1 F-loci *of the second kind* conjugate under $G_{2\cdot 2^4}$. These F-loci of the second kind are on W_2 and each is invariant under $I_{12\ldots 6}$ which defines W_2. On the other hand the $2\cdot 2^4$ loci of the first kind are paired by $I_{12\ldots 6}$, π_1 with π_{23456} and π_{123} with π_{456}, and the intersection of each of the 2^4 pairs is a locus on W_2. The product of the members of a pair is a quadric on P_6^3. Quadrics on P_6^3 map W_2 into the Kummer quartic surface, the F-loci of the first kind of W_2 contributing the 16 tropes of the Kummer surface, and the F-loci of the second kind contributing the 16 nodes of the Kummer surface. Each locus of one kind is on $2p+2=6$ loci of the other kind.

Similarly, W_3 has three kinds of F-loci each forming a conjugate set of $2\cdot 2^6$, except for 2^6 of the third kind which are on W_3. Those of the first kind have dimension 0 or 4; of the second kind, dimension 1 or 3; of the third kind, dimension 2. The mapping system which converts W_3 into the generalized hyperelliptic Kummer 3-way of Klein and Wirtinger is a system of cubic spreads with nodes at P_8^5, the F-spaces of the first kind of dimension 0, and simple points on the F-spaces of the second kind of dimension 1. Thus it turns out that the hyperelliptic Kummer 3-way in S_7 has, in addition to 64 singular points and 64 singular S_6's, also a set of 64 singular S_3's.

Generically, then, W_p has p kinds of pairs of F-loci with the following incidence relations: Each pair of F-loci of the jth kind, $(j=1, \cdots, p)$, contains $2p+2$ pairs of F-loci of the $(j+1)$th kind and is contained by $2p+2$ pairs of F-loci of the $(j-1)$th kind. Thus the hyperelliptic Kummer p-way K_p has, in addition to the 2^{2p} singular points and 2^{2p} singular spaces given by the classic theta function theory, $p-1$ sets of 2^{2p} singular linear spaces of intermediate dimension with the incidence relations just mentioned.

The mapping system which converts W_p into K_p, a system of spreads of order p with $(p-1)$-fold points at P_{2p+2}^{2p-1}, has essential importance. If its dimension is less than 2^p-1, then W_p is mapped not into K_p but rather into some projection of K_p (I have shown elsewhere that the theta squares are represented by members of this system). When $p=2$, quadrics in S_3 on P_6^3 have the dimension $9-6=2^2-1$; when $p=3$, cubics in S_5 with nodes at P_8^5 have the dimension $C_{8,5}-8C_{6,5}-1=2^3-1$; when $p=4$ the argument is not so simple. Quartics in S_7 have the dimension $C_{11,7}-1=330-1$; the 10 triple points contribute $10C_{9,7}=360$ conditions. However, if p and q are triple points of the quartic $(\alpha x)^4=0$, then $(\alpha p)^2(\alpha x)^2\equiv 0$ and $(\alpha q)^2(\alpha x)^2\equiv 0$. Thus the condition $(\alpha p)^2(\alpha q)^2=0$ is counted twice. The revised dimension is now $330-1-360+45=2^4-2$, and is not the expected 2^4-1. The method of counting conditions just used is quite all right for 8 triple points, since $C_{11,7}-1-8.36+C_{8,2}=C_{8,4}-1$ and the system has the equation $\sum a_{ijkl}x_ix_jx_kx_l=0$, $(i, j, k, l=0, \cdots, 7)$. If one imposes on this system the further conditions that y and z be triple points (28 conditions for each with one dependence for the pair) there is certainly no easy way to see algebraically that the apparently independent 55 conditions are still dependent. Naturally such complications would increase with increasing p. If, however, the coordinate system with reference to N^{2p-1} on P_{2p+2}^{2p-1} be employed, the members of the mapping system are represented by symmetric forms of order $2p-1$ in p sets of binary variables. One may then show with relative ease that the linear system Σ of order p with $(p-1)$-fold points at P_{2p+2}^{2p-1} has the dimension 2^p-1 and that it contains all the F-loci of the jth kind and dimension $j-1$ to the multiplicity $p-j$, $(j=1, \cdots, p)$. Furthermore, the dimension of the sub-system of Σ which con-

tains a pair of F-loci (paired under $I_{12\cdots 2p+2}$) is also readily obtained so that the dimensions of the intermediate singular linear spaces of K_p are likewise known.

Let us pass to a second series of loci on hyperelliptic K_p which is suggested by W_p. There is a familiar planar theorem to the effect that if two generically placed rational envelopes of class m and n are put into generic one-to-one correspondence, the locus of the intersection of corresponding lines is a rational curve of order $m+n$. We say that the two envelopes *generate* the rational curve. The curve is then *perspective* to either envelope, that is, like named point of the one and line of the other are incident. This leads to a rough classification of rational curves of given order. If, for example, a rational septimic has a perspective cubic (necessarily unique), it is generated by this cubic and a perspective quartic. The septimic may, however, be specialized to the point of having a perspective conic. It then has no proper perspective cubic and must be generated by the perspective conic and a perspective quintic. The septimic may even have a perspective point, necessarily a six-fold point. It is then generated by this point and a perspective sextic. Thus the septimic must be one of three types, say type (3, 4), type (2, 5), or type (1, 6).

There is a somewhat analogous classification of hyperelliptic curves $H_p{}^{p+2}$. If two such curves $H_q{}^{q+2}$, $H_r{}^{r+2}$ have a common center O, a line through O meets them in two pairs of points with a common harmonic pair. The locus of this pair as the line turns about O is a hyperelliptic H_{q+r+1}^{q+r+3} on the branch points of $H_q{}^{q+2}$, $H_r{}^{r+2}$, with center O, and with branch points at the common points of these two curves. We shall say again that H_{q+r+1}^{q+r+3} is generated by $H_q{}^{q+2}$, $H_r{}^{r+2}$. The condition that $H_q{}^{q+2}$ should serve as a generator of $H_p{}^{p+2}$ is that the two have the same center O and that either be on the branch points of the other. Analytically the condition is

$$f_p g_{q+2} - 2f_{p+1}g_{q+1} + f_{p+2}g_q \equiv 0.$$

This imposes $q+p+3$ conditions of which $3q+5$ can be absorbed by the three forms g, leaving $p-2q-2$ conditions on the three forms f. Thus it is $p-2$ conditions that the branch points of $H_p{}^{p+2}$ be on a conic, $p-4$ conditions that they be on a cubic through O, $p-6$ that they be on a quartic with node at O, etc.

Hence there is on W_p a locus $V_2(p)$ whose points represent H_p^{p+2}'s with branch points on a conic, a locus $V_4(p)$ whose points represent H_p^{p+2}'s with branch points on a cubic, and thus a chain of loci $V_2(p)$, $V_4(p)$, $V_6(p)$, $\cdots$ is defined on W_p.

The first of these, $V_2(p)$, has a number of the outstanding properties of the original Weddle W_2 with which it coincides when $p=2$. I give one example. By proper choice of the line of reference y_0 opposite O, the conic on the branch points of H_p^{p+2} can be given the form, $y_0^2-(at)^2=0$. The above identity is now $f_p\cdot(at)^2-f_{p+2}=0$, whence the fundamental $(2p+2)$-ic is $(\omega t)^{2p+2}=f_{p+1}^2-f_p^2\cdot(at)^2=0$. Thus, for the branch parameters t_i, we have $f_{p+1}(t_i)/f_p(t_i)=\sqrt{(at_i)^2}$, and the formula (A) given earlier becomes now

$$(at)^{2p-1}=\sum_{i=1}^{i=2p+2}\frac{\sqrt{(at_i)^2}\cdot(tt_i)^{2p-1}}{\omega'(t_1)}.$$

This for variation of the quadratic $(at)^2$ yields the ∞^2 points of $V_2(p)$. It may be shown that the point given remains on $V_2(p)$ for arbitrary choice of signs of the $2p+2$ radicals. If the sign of the first radical be changed a new point of $V_2(p)$ is obtained which is on a line with the original point and the singular point p_1. Hence we have the following theorem.

The line on a singular point of $V_2(p)$ and a point of $V_2(p)$ cuts $V_2(p)$ again; if then a point x of $V_2(p)$ is projected from the $2p+2$ singular points of $V_2(p)$ into $2p+2$ new points of $V_2(p)$, if these new points are similarly projected into further points, etc., a system of 2^{2p+1} points on $V_2(p)$ is obtained which is closed under such projection.

For $p=2$ and $V_2(p)=W_2$ this theorem has been given by H. F. Baker.

There are on P_{2p+2}^{2p-1} ∞^4 normal elliptic curves E^{2p}. The mapping system Σ cuts such an E^{2p} in a g_1^2 which has four branch points on W_p. If E^{2p} is binodal and therefore degenerate, the locus of the nodes is $V_2(p)$; if it is not so degenerate, the locus of the ∞^4 branch points is $V_4(p)$. This exhausts the possibilities connected with the family of elliptic curves, and I have not thus far found a generalization of this family which will serve for the isolation of $V_6(p)$, $V_8(p)$, etc.

In conclusion I wish to call attention to a third system of loci on W_p which arises from extensions and generalizations of a theorem due to two of my associates of other days at Johns Hopkins, Professors Morley and Conner. There is a fairly old theorem relating to the Kummer surface K_2, with many interesting contacts, which states that the section of K_2 by a generic plane is a generic quartic curve. This might be expected since K_2 has three absolute constants and the plane adds three more. However, W_2, the Weddle quartic surface, also has three absolute constants, but its generic section, Q, as Morley and Conner pointed out twenty-five years ago (American Journal of Mathematics, vol. 31 (1909), p. 263), is not a generic quartic curve. They determine the invariant of degree six which Q satisfies. They explain its geometric character by using a pencil of Weddle's as follows. The W_2 is localized by its hexad of nodes P_6^3 on N^3, that is, by a binary sextic, $(\omega t)^6$. The pencil of binary sextics, $\lambda(\omega t)^6+[(\mu t)^3]^2$, determines a pencil of W_2's, and in particular the member determined by $[(\mu t)^3]^2$ contains the plane $(\mu t)^3$, trisecant to N^3, as a part. Thus the quartic section Q of W_2 by this plane is the same as the section of each W_2 of the pencil. But each W_2 contains the 15 lines joining its hexad of nodes. Hence the ∞^1 sextics $\lambda(\omega t)^6+[(\mu t)^3]^2$ determine on N^3 ∞^1 six-points whose complete figures are cut by a plane in the quartic curve Q containing ∞^1 configurations $(15_3, 20_4)$.

This notion of a *pencil* of W_2's cannot be extended. The method which I am about to give not only yields additional information with respect to the section of W_2 but also applies to the section of W_p in S_{2p-1} by a $(p+1)$-secant S_p of N_{2p-1}. Taking $(\omega t)^6=f_3^2-f_2f_4$, we have already indicated the roots $t_1, \cdots, t_6$ of $(\omega t)^6$, and the roots r_1, r_2 of f_2. Let also s_1, s_2, s_3 be the roots of f_3, so that

$$f_3 = c(ts_1)(ts_2)(ts_3).$$

For variation of the constant factor c in f_3 we have in

$$(\omega t)^6 - f_3^2 = -f_2f_4$$

a pencil of binary sextics and f_2 is a duad in some hexad of the pencil. The seventh powers of the 9 linear forms, (tt_i), (ts_j), are connected by a linear identity, and the polar of f_4 as to this identity yields the following identity:

$$\sum_{i=1}^{i=6} \frac{(tt_i)^2 \cdot f_4(t_i)}{f_3(t_i) \cdot \omega'(t_i)} + \sum_{j=1}^{j=3} \frac{(ts_j)^2 \cdot f_4(s_j)}{\omega(s_j) \cdot f_3'(s_j)} \equiv 0.$$

Since $f_4(t_i)/f_3(t_i) = f_3(t_i)/f_2(t_i)$, we see by comparison with (A) that

$$(C) \qquad -(\alpha t)^3 = \sum_{j=1}^{j=3} \frac{(ts_j)^2 \cdot f_4(s_j)}{\omega(s_j) \cdot f_3'(s_j)}.$$

Since also $f_4(s_j)/\omega(s_j) = -1/f_2(s_j)$,

$$(D) \qquad (\alpha t)^3 = \sum_{j=1}^{j=3} \frac{(ts_j)^3}{f_2(s_j) \cdot f_3'(s_j)}.$$

Hence, for given trisecant plane $\pi = s_1 s_2 s_3$ of N^3, and for a selected duad f_2, in some hexad of the pencil above, there is a point $(\alpha t)^3$ on W_2 which, according to (C), is on the plane π, and, according to (B), is on the bisecant $r_1 r_2$ of N^3 determined by f_2. This proves the Morley-Conner theorem that the section Q of W_2 by π contains the inscribed configurations which they observed.

But the three-term expression (D) contributes new information. It contains the variable duad f_2 of the pencil in the denominator. Hence the point of Q on the left is the quadratic transform of the point $f_2(t)$ in a plane π'. This yields the following theorem.

If the hexads of the pencil, $(\omega t)^6 - f_3^2 = 0$, be plotted as hexads of tangents of a norm-conic N^2 in a plane π', the locus of the vertices of the complete hexagons is a quintic curve with three nodes whose joins are the three tangents $f_3 = 0$ of N^2; and the transform of this quintic by a quadratic transformation with F-points at the nodes is a section Q of W_2 by the plane $f_3 = 0$ with the configurations indicated.

These results can be generalized immediately to apply to the curves in which W_p is cut by the $(p+1)$-secant S_p's of N^{2p-1} determined by f_{p+1}. They can also be extended to cover the cases where the $(p+1)$-secant S_p passes through some of the points of P_{2p+2}^{2p-1}. Suppose, for example, that $j+2$ of the branch points, determined by $\lambda_{j+2}(t) = 0$, are on the line $y_0 = 0$. Then λ_{j+2} is a factor of f_{p+2}, of $(\omega t)^{2p+2}$, and therefore of f_{p+1}. If we set

$$(\omega t)^{2p+2} = \mu_{2p-j} \cdot \lambda_{j+2}, \quad f_{p+2} = g_{p-j} \cdot \lambda_{j+2}, \quad f_{p+1} = g_{p-j-1} \cdot \lambda_{j+2},$$

the pencil, analogous to the above pencil of sextics, becomes

$$\mu_{2p-j} - g^2_{p-j-1} \cdot \lambda_{j+2} = -f_p \cdot g_{p-j}.$$

For $j=-2$ we have the generic case mentioned above. For $j=-1$, $j=0$ we have sections of W_p by S_p's on one or two points of P^{2p-1}_{2p+2}. For $j=1, 2, \cdots, p$ we have curves on W_p which also lie on the F-spaces of the first, second, $\cdots$, pth kind.

I may say finally that the importance of W_p lies in the information which it furnishes with respect to those properties which differentiate the hyperelliptic K_p from the K_p attached to more general theta functions.

The study of this manifold has had for me a peculiar fascination because of the demand it makes on geometric ingenuity. General methods of attack have been quite uniformly fruitless because of one or another special characteristic of the manifold. Doubtless we all are accustomed to seeing the waste basket claim the major part of our labour. In this respect at least I may justly assert that W_p is the most outstanding topic I have ever studied.

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ELEMENTARY DERIVATION OF THE EQUIVALENCE OF MASS AND ENERGY*

BY ALBERT EINSTEIN

The special theory of relativity grew out of the Maxwell electromagnetic equations. So it came about that even in the derivation of the mechanical concepts and their relations the consideration of those of the electromagnetic field has played an essential role. The question as to the independence of those relations is a natural one because the Lorentz transformation, the real basis of the special relativity theory, in itself has nothing to do with the Maxwell theory and because we do not know the extent to which the energy concepts of the Maxwell theory can be maintained in the face of the data of molecular physics. In the following considerations, except for the Lorentz transformation, we will depend only on the assumption of the conservation principles for impulse and energy.

We begin by making plausible the expressions for impulse and energy of the material particle in the well known way. The fundamental invariant of the Lorentz transformation is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2,$$

or

$$ds = dt(1 - u^2)^{1/2},$$

where

$$u^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = u_1^2 + u_2^2 + u_3^2.$$

If one divides the components of the contravariant vector (dt, dx, dy, dz) by ds, one obtains the vector

$$\left(\frac{1}{(1-u^2)^{1/2}}, \frac{u_1}{(1-u^2)^{1/2}}, \frac{u_2}{(1-u^2)^{1/2}}, \frac{u_3}{(1-u^2)^{1/2}}\right).$$

* The Eleventh Josiah Willard Gibbs Lecture, delivered at Pittsburgh, December 28, 1934, under the auspices of this Society, at a joint meeting of this Society, the American Physical Society, and Section A of the A.A.A.S.

Let (dt, dx, dy, dz) belong to the world-line of a material particle of mass m. We obtain a vector connected with the motion of the latter by multiplying by m the *four-vector* of velocity that we have just written down. We thus have

$$(\eta^\sigma) = \left(\frac{m}{(1-u^2)^{1/2}}, \frac{mu_i}{(1-u^2)^{1/2}}\right),$$

where the index i runs from 1 to 3.

Neglecting the third power of the velocity, we can express the components of this vector by

$$(\eta^\sigma) = (m + \frac{1}{2}mu^2,\ mu_i).$$

The *space* components of (η^σ) express in this approximation the components of the impulse in classical mechanics, while the *time* component, aside from the additive constant m, expresses the kinetic energy of the material point.

If one goes back to the exact expression for (η^σ), it is therefore natural to take

$$\frac{mu_i}{(1-u^2)^{1/2}}$$

as the impulse and

$$m\left(\frac{1}{(1-u^2)^{1/2}} - 1\right)$$

as the kinetic energy of the particle. However, how is one to interpret the time component $m/(1-u^2)^{1/2}$ itself, the expression for which after all is the really significant one? Here it is natural to give it directly the meaning of energy, hence to ascribe to the mass-point in a state of rest the *rest-energy* m (with the usual time unit, mc^2).

Of course, this derivation cannot pretend to be a proof since in no way is it shown that this impulse satisfies the impulse-principle and this energy the energy-principle if several particles of the same kind interact with one another; it may be a priori conceivable that in these conservation-principles different expressions of the velocity are involved.

Furthermore, it is not perfectly clear as to what is meant in

speaking of the *rest-energy*, as the energy is defined only to within an undetermined additive constant; in connection with this, however, the following is to be remarked. Every system can be looked upon as a material point as long as we consider no processes other than changes in its translation velocity as a whole. It has a clear meaning, however, to consider changes in the rest-energy in case changes are to be considered other than mere changes of translation velocity. The above interpretation asserts, then, that in such a transformation of a material point its inertial mass changes as the rest-energy; this assertion naturally requires a proof.

What we will now show is the following. If the principles of conservation of impulse and energy are to hold for all coordinate systems which are connected with one another by the Lorentz transformations, then impulse and energy are really given by the above expressions and the presumed equivalence of mass and rest-energy also exists.

We start from some simple kinematic consequences of the Lorentz transformation:

$$t = \frac{t' + vx'}{(1 - v^2)^{1/2}}, \qquad x = \frac{x' + vt'}{(1 - v^2)^{1/2}}, \qquad y = y', \qquad z = z',$$

where v is the relative velocity of the coordinate systems K and K'. The same relations hold for the differentials dx, etc. By suitable division one obtains the law of transformation of the velocities:

$$u_1 = \frac{u_1' + v}{1 + u_1' v}, \qquad u_2 = \frac{u_2' (1 - v^2)^{1/2}}{1 + u_1' v}, \qquad u_3 = \frac{u_3' (1 - v^2)^{1/2}}{1 + u_1' v}.$$

From this one obtains

$$u^2 = \frac{u'^2 + 2u_1' v + v^2 - u_2'^2 v^2 - u_3'^2 v^2}{(1 + u_1' v)^2},$$

and

$$\frac{1}{(1 - u^2)^{1/2}} = \frac{1 + u_1' v}{(1 - u'^2)^{1/2}(1 - v^2)^{1/2}},$$

as well as

$$\frac{u_1}{(1-u^2)^{1/2}} = \frac{u_1' + v}{(1-u'^2)^{1/2}(1-v^2)^{1/2}},$$

$$\frac{u_2}{(1-u^2)^{1/2}} = \frac{u_2'}{(1-u'^2)^{1/2}}, \qquad \frac{u_3}{(1-u^2)^{1/2}} = \frac{u_3'}{(1-u'^2)^{1/2}}.$$

We introduce now for the following the concept of the *particle-pair*. By this we understand two material points with equal and opposite velocities referred to K' (and later to be chosen of equal mass). The two particles are designated by the indices $+$ or $-$. Hence $u_+' = u_-'$, $u_{1+}' = -u_{1-}'$, etc. Applying to these our transformation equations after addition, we get

$$(1)\quad \begin{aligned} &\frac{1}{(1-u_+^2)^{1/2}} + \frac{1}{(1-u_-^2)^{1/2}} = \frac{2}{(1-u'^2)^{1/2}(1-v^2)^{1/2}},\\ &\frac{u_{1+}}{(1-u_+^2)^{1/2}} + \frac{u_{1-}}{(1-u_-^2)^{1/2}} = \frac{2v}{(1-u'^2)^{1/2}(1-v^2)^{1/2}},\\ &\frac{u_{2+}}{(1-u_+^2)^{1/2}} + \frac{u_{2-}}{(1-u_-^2)^{1/2}} = 0,\\ &\frac{u_{3+}}{(1-u_+^2)^{1/2}} + \frac{u_{3-}}{(1-u_-^2)^{1/2}} = 0. \end{aligned}$$

The sums on the left sides of these equations depend, therefore, only on the velocity u' of the pair referred to the special system K' and on the relative velocity v of K' with respect to K, not however on the direction in which the particles are moving.

We remark that the equations (1) can be derived more clearly if one considers directly the transformation for the sum of the four-vectors of the velocities of a particle-pair. I have chosen the above representation, however, because the conservation laws indicate the use of this 3-dimensionally inhomogeneous manner of writing.

We now go to the actual considerations. We assume that impulse and energy of a material point are given by expressions of the form

$$I_\nu = mu_\nu F(u), \qquad E = E_0 + nG(u), \qquad (\nu = 1, 2, 3),$$

where F and G are universal, even functions of the velocity u, which vanish for $u = 0$. Then $mG(u)$ is the kinetic energy, E_0 the

rest-energy of the material point, m the *rest-mass* or, simply, the mass. It is here assumed that impulse and energy of the *mass-point* are independent of direction of motion and of the orientation of the *mass-point* relative to the velocity. It is further assumed that in impulse and energy the *same* mass-constant m occurs, for which, however, we shall find later a partial justification.

We consider now the elastic eccentric collision between two particles of equal mass. One can always choose the coordinate-system K' so that, referred to the latter, the velocities of the masses before the collision are equal in magnitude and opposite in direction; how are the velocities after the collision constituted with reference to K'? If the velocities after the collision were not likewise equal and opposite, the impulse law would be violated. If the common velocity of both masses after the collision were not equal in magnitude to that before the collision, in case the collision is elastic, the energy law would be violated. This holds independently of the particular law of dependence of impulse and energy on the velocity. The collision, therefore, changes only the direction of motion of the two mass-points referred to K'. We can express this briefly as follows. A particle-pair before the collision is transformed by the latter again into a particle-pair of the same velocity u'.

The right side of (1) is therefore not changed by the collision. It follows then, from (1), that, referred to K, we have for the states before and after the collision the equations:

$$(2)\qquad \begin{aligned} \frac{1}{(1-u_+^2)^{1/2}}+\frac{1}{(1-u_-^2)^{1/2}} &= \frac{1}{(1-\bar{u}_+^2)^{1/2}}+\frac{1}{(1-\bar{u}_-^2)^{1/2}}, \\ \frac{u_{i+}}{(1-u_+^2)^{1/2}}+\frac{u_{i-}}{(1-u_-^2)^{1/2}} &= \frac{\bar{u}_{i+}}{(1-\bar{u}_+^2)^{1/2}}+\frac{\bar{u}_{i-}}{(1-\bar{u}_-^2)^{1/2}}. \end{aligned}$$

Barred quantities are those which refer to the state after the collision. These equations, which are valid in general for elastic collisions of equal masses, have the form of conservation equations; it may therefore be taken for granted that no other symmetrical or anti-symmetrical functions of the velocity-components exist which in the present case of the elastic collision of two identically constituted material points give anything analogous. Consequently we shall have to regard

$$\frac{mu_i}{(1-u^2)^{1/2}} \tag{3}$$

as impulse and

$$m\left(\frac{1}{(1-u^2)^{1/2}}-1\right) \tag{4}$$

as the kinetic energy of a particle.*

We now turn to the proof of the assertion that the mass is equal to the *rest-energy*. For the total energy E of a moving particle we shall have to take

$$E = E_0 + m\left(\frac{1}{(1-u^2)^{1/2}}-1\right), \tag{4a}$$

where we imagine that E_0 (rest-energy) and m can be changed in the case of interactions of material points that are not *elastic*.

We now consider the inelastic collision between two particles of equal mass and equal rest-energy, which before the collision again form a *particle-pair* with respect to K' (velocities equal and opposite). We assume here for simplicity that the internal changes which the material points suffer in the collision are equal to each other. From the impulse principle referred to K' it follows as above that the final velocities of the two particles must be equal and opposite ($\bar{u}_+' = -\bar{u}_-'$). The energy law gives then, referred to K' and K, respectively,

$$2E_0 + 2m\left(\frac{1}{(1-u'^2)^{1/2}}-1\right) = 2\overline{E}_0 + 2\bar{m}\left(\frac{1}{(1-\bar{u}'^2)^{1/2}}-1\right),$$

$$2E_0 + m\left(\frac{1}{(1-u_+^2)^{1/2}}-1\right) + m\left(\frac{1}{(1-u_-^2)^{1/2}}-1\right)$$
$$= 2\overline{E}_0 + \bar{m}\left(\frac{1}{(1-\bar{u}_+^2)^{1/2}}-1\right) + \bar{m}\left(\frac{1}{(1-\bar{u}_-^2)^{1/2}}-1\right).$$

As the points form a pair before and after the collision, one can write the last equation on the basis of (1) in the form

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* This must naturally vanish for $u=0$; for it is defined as the expenditure of energy necessary to impart the velocity u to the particle initially at rest (without internal change).

$$E_0 - m + \frac{m}{(1-u'^2)^{1/2}(1-v^2)^{1/2}}$$

$$= \bar{E}_0 - \bar{m} + \frac{\bar{m}}{(1-\bar{u}'^2)^{1/2}(1-v^2)^{1/2}}.$$

The first equation we write analogously in the form

$$(5) \qquad E_0 - m + \frac{m}{(1-u'^2)^{1/2}} = \bar{E}_0 - \bar{m} + \frac{\bar{m}}{(1-\bar{u}'^2)^{1/2}}.$$

Multiplying the last equation by $1/(1-v^2)^{1/2}$ and subtracting from the previous one we get

$$[(\bar{E}_0 - E_0) - (\bar{m} - m)]\left(\frac{1}{(1-v^2)^{1/2}} - 1\right) = 0,$$

or

$$(6) \qquad \bar{E}_0 - E_0 = \bar{m} - m.$$

The rest-energy changes, therefore, in an inelastic collision (additively) like the mass. As the former, from the nature of the concept, is determined only to within an additive constant, one can stipulate that E_0 should vanish together with m. Then we have simply

$$E_0 = m,$$

which states the principle of equivalence of inertial mass and rest-energy.

If we apply the impulse law to the x-component, it follows (for an inelastic collision) that

$$m\frac{u_{+1}}{(1-u_+^2)^{1/2}} + m\frac{u_{-1}}{(1-u_-^2)^{1/2}} = \bar{m}\frac{\bar{u}_{+1}}{(1-\bar{u}_+^2)^{1/2}} + \bar{m}\frac{\bar{u}_{-1}}{(1-\bar{u}_-^2)^{1/2}},$$

or by the application of the second of equations (1), for the state before and after the collision,

$$\frac{m}{(1-u^2)^{1/2}} = \frac{\bar{m}}{(1-\bar{u}^2)^{1/2}}.$$

The same relation follows also from equations (5) and (6) which were obtained from the energy principle. If, from the

beginning, we had provided the expression for the impulse with a mass-constant different from that of the energy, these considerations would show that the impulse-mass changes in an inelastic collision like the energy-mass. This is a partial justification for setting both mass-constants equal to each other.

The result of this consideration is therefore as follows. If for collisions of material points the conservation laws are to hold for an arbitrary (Lorentz) coordinate-system, the well known expressions for impulse and energy follow, as well as the validity of the principle of equivalence of mass and rest-energy.

Professor G. D. Birkhoff has called my attention to the fact that in the book which he has written in collaboration with Professor R. E. Langer, *Relativity and Modern Physics*, quite similar considerations are given concerning collisions of particles, as well as concerning impulse and energy. In spite of this, I believe that the present derivations merit a certain amount of interest.

Thus, in the book just mentioned, essential use is made of the concept of *force*, which in the relativity theory has no such direct significance as it has in classical mechanics. This is connected with the fact that, in the latter, the force is to be considered as a given function of the coordinates of all the particles, which is obviously not possible in the relativity theory. Therefore I have avoided introducing the force concept.

Furthermore, I was concerned with avoiding making any assumption concerning the transformation character of impulse and energy with respect to a Lorentz transformation.

INSTITUTE FOR ADVANCED STUDY

A NOTE ON TAYLOR'S THEOREM

BY A. F. MOURSUND

Let the function $f(x)$ be such that $f^{(n)}(a) \equiv d^n f(x)/dx^n$ at $x=a$ exists; then, for $|h|$ sufficiently small, we can write

$$(1)\quad f(a+h)=f(a)+h\,f'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^n}{n!}f^{(n)}(a)+w(a,\,h).$$

It is well known that $w(a, h)=o(h^n)$ as $h\to 0$,* and the more precise result that $|\,w(a,\,h)\,| \leqq |\,h^n\,|\,v(a,\,h)$, where $v(a, h)$ is the least upper bound for $0<|t|<|h|$ of

$$\left|\frac{f^{(n-1)}(a+t)-f^{(n-1)}(a)}{t}-f^{(n)}(a)\right|$$

is given by S. Pollard.†

In this note we are concerned primarily with the behavior, as $h\to 0$, of derivatives with respect to h of the function $w(a, h)$. The point a being fixed, we designate the ith such derivative, $i\geqq 0$, by $d^i w(a, h)/dh^i$. Our theorem, a generalization of Pollard's theorem, is given below.

THEOREM. *If $f(x)$ is such that $f^{(n)}(a)$ exists, then for $i=0, 1, 2, \cdots, n-1$, and $|h|$ sufficiently small*

$$\left|\frac{d^i}{dh^i}\,w(a,\,h)\right| \leqq \frac{|\,h^{n-i}\,|}{(n-i)!}\,v(a,\,h).$$

PROOF. Since

$$\frac{d^i}{dt^i}\,f(a+t) \equiv \frac{d^i}{dx^i}\,f(x)\Big]_{x=a+t} \equiv f^{(i)}(a+t),$$

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* See E. W. Hobson, *The Theory of Functions of a Real Variable*, vol. 1, 3d ed., pp. 368–370. We use here the more restrictive of the two definitions given by Hobson for $f^{(n)}(x)$. The existence of $f^{(n)}(a)$ then insures the existence and continuity in an open interval containing a of all derivatives of lower order.

† S. Pollard, *On the descriptive form of Taylor's theorem*, Cambridge Philosophical Society Proceedings, vol. 23 (1926–27), pp. 383–385. Pollard's proof seems only to establish the less sharp result $|w(a, h)| \leqq n|h^n|v(a, h)$.

we see, upon writing t for h in (1) and differentiating, that (i) for $i<n$,

$$\frac{d^i}{dt^i}\, w(a, t) = o(1), \text{ as } t \to 0,$$

which insures that for $|t|$ sufficiently small and $j=1, 2, \cdots, n-1$,

$$\int_0^t \frac{d^j}{dt^j}\, w(a, t)dt = \frac{d^{j-1}}{dt^{j-1}}\, w(a, t);$$

and that (ii) for $|h|$ sufficiently small and $|t|<|h|$,

$$\left| \frac{d^{n-1}}{dt^{n-1}}\, w(a, t) \right| = \left| t\left[\frac{f^{(n-1)}(a+t) - f^{(n-1)}(a)}{t} - f^{(n)}(a)\right]\right| \leqq |t|\, v(a, h).$$

We have then for $|h|$ sufficiently small

$$\begin{aligned}
&\left| \frac{d^i}{dh^i}\, w(a, h) \right| \\
&= \left| \int_0^h dt_{n-i-2} \int_0^{t_{n-i-2}} dt_{n-i-3} \cdots \int_0^{t_1} \frac{d^{n-1}}{dt^{n-1}}\, w(a, t)dt \right| \\
&\leqq \left| \int_0^h dt_{n-i-2} \int_0^{t_{n-i-2}} dt_{n-i-3} \cdots \int_0^{t_1} \left| \frac{d^{n-1}}{dt^{n-1}}\, w(a, t) \right| dt \right| \\
&\leqq \frac{|h^{n-i}|}{(n-i)!}\, v(a, h).
\end{aligned}$$

Since $v(a, h)=o(1)$ as $h \to 0$, it follows from our theorem that for $i=0, 1, 2, \cdots, n-1$,

$$\frac{d^i}{dh^i}\, w(a, h) = o(h^{n-i}), \text{ as } h \to 0.*$$

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* For $i>0$ the result given here can be obtained from that for $i=0$ by comparing the expansion analogous to (1) of $f^{(i)}(a+h)$ with the equation obtained by differentiating (1) i times.

A THEOREM ON ANALYTIC FUNCTIONS OF A REAL VARIABLE

BY R. P. BOAS, JR.

1. *Introduction.* Let $f(x)$ be a function of class C^∞ on $a \leqq x \leqq b$. At each point x of $[a, b]$ we form the formal Taylor series of $f(x)$,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} (t - x)^k.$$

This series has a definite radius of convergence, $\rho(x)$, zero, finite, or infinite, given by $1/\rho(x) = \overline{\lim}_{k\to\infty} \left| f^{(k)}(x)/k! \right|^{1/k}$. The function $f(x)$ is said to be analytic at the point x if the Taylor development of $f(x)$ about x converges to $f(t)$ over a neighborhood $|x-t| < c$, $c>0$, of the point; $f(x)$ is analytic in an interval if it is analytic at every point of the interval.

Pringsheim stated the following theorem.*

THEOREM A. *If there exists a number $\delta>0$ such that $\rho(x) \geqq \delta$ for $a \leqq x \leqq b$, $f(x)$ is analytic in $[a, b]$.*

However, Pringsheim's proof of the theorem is not rigorous. The purpose of this note is to establish this theorem, and, in connection with the proof, a companion theorem of considerable interest in itself.

THEOREM B. *If $\rho(x)>0$ for $a \leqq x \leqq b$ (that is, if the Taylor development of $f(x)$ about each point converges in some neighborhood of the point), the points at which $f(x)$ is not analytic form at most a nowhere dense closed set.*

Theorem B is, in a certain sense, the best possible, since by a theorem of H. Whitney† there exist functions satisfying the

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* A. Pringsheim, *Zur Theorie der Taylor'schen Reihe und der analytischen Funktionen mit beschränkten Existenzbereich*, Mathematische Annalen, vol. 42 (1893), p. 180.

† H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of this Society, vol. 36 (1934), pp. 63–89. I am indebted to Dr. Whitney for calling my attention to this paper.

conditions of Theorem B and having the points of an arbitrary nowhere dense closed set as singular points.

Theorem B can also be stated in the following equivalent form.

THEOREM B′. *If $f(x)$ is of class C^∞ on $[a, b]$ and analytic at no point of $[a, b]$, there must exist an everywhere dense set of points, G, on $[a, b]$ such that the Taylor development of $f(x)$ about each point of G is divergent.*

We shall need the following lemma.*

LEMMA. *If H is a perfect point set on the interval $[\alpha, \beta]$, and if $H=\sum_{n=0}^{\infty} H_n$, where the H_n are enumerable in number, there exist a value n_0 of n and a sub-interval $[\alpha_0, \beta_0]$ such that H_{n_0} is dense in $H \cdot [\alpha_0, \beta_0]$.*

2. *Proof of Theorem* B. For each x in $[a, b]$,

$$\frac{1}{\rho(x)} = \overline{\lim_{n\to\infty}} \left| \frac{f^{(n)}(x)}{n!} \right|^{1/n} < \infty .$$

This implies that there exists a finite function $\mu(x)$ such that

$$\left| \frac{1}{n!} f^{(n)}(x) \right|^{1/n} \leqq \mu(x), \qquad (n = 1, 2, \cdots),$$

or,

$$| f^{(n)}(x) | \leqq n![\mu(x)]^n, \qquad (n = 1, 2, \cdots).$$

Let E_k be the set (not necessarily non-empty) of points x such that

$$k \leqq \mu(x) < k + 1, \qquad (k = 0, 1, 2, \cdots).$$

It is clear that $[a, b]=\sum_{k=0}^{\infty} E_k$. By the lemma, there is a sub-interval $[\alpha, \beta]$ and an integer k_0 such that E_{k_0} is dense in $[\alpha, \beta]$. For every point of $E_{k_0} \cdot [\alpha, \beta]$,

$$(1) \qquad | f^{(n)}(x) | \leqq n![\mu(x)]^n < n!(k_0 + 1)^n, \qquad (n = 1, 2, \cdots).$$

For every point of $[\alpha, \beta] \cdot C(E_{k_0})$, (1) holds by continuity. That

* See Lebesgue, *Leçons sur l'Intégration*, 2d ed., 1928, p. 203. See also S. Banach, *Théorie des Opérations Linéaires*, 1932, p. 14 (Theorem 2).

is, (1) holds uniformly in $[\alpha, \beta]$.* But this is a well known sufficient condition for $f(x)$ to be analytic in $[\alpha, \beta]$. The same reasoning applies to any sub-interval of $[a, b]-[\alpha, \beta]$; thus in any sub-interval there is a further sub-interval in which $f(x)$ is analytic. The points at which $f(x)$ is not analytic thus form a nowhere dense set, which is obviously closed.

3. *Proof of Theorem* A. Assume the theorem false; we shall obtain a contradiction. We have, then, a non-empty set H of points where $f(x)$ is not analytic, and by Theorem B, H is closed and nowhere dense.

We first show that H is perfect. Suppose that H contained an isolated point X. The function $f(x)$ is continuous with all derivatives at X; $f(x)$ is analytic in each of the intervals $X-h<x<X$ and $X<x<X+h$, for some $h>0$, and can be extended analytically across the point X in both directions. It follows immediately that $f(x)$ is analytic at X, so that X is not a singular point. H being perfect, from now on we shall confine our attention to an interval $[a_1, b_1]$ such that $b_1-a_1<\delta/4$ and $[a_1, b_1]$ contains a perfect subset E of H.

Since by hypothesis

$$\overline{\lim_{n\to\infty}} \left| \frac{f^{(n)}(x)}{n!} \right|^{1/n} = \frac{1}{\rho(x)} \leqq \frac{1}{\delta}$$

for every point of $[a_1, b_1]$, it follows that for each x in E there is an integer N_x such that

$$\left| \frac{1}{n!} f^{(n)}(x) \right|^{1/n} < \frac{2}{\delta}, \qquad (n \geqq N_x);$$

hence

$$(2) \qquad | f^{(n)}(x) | \leqq n!\lambda^n, \qquad (\lambda = 2/\delta,\ n \geqq N_x).$$

Let E_k be the set of points of E for which $N_x=k$. By the lemma, there exist a sub-interval $[\alpha, \beta]$ and a value k_0 of k such that E_{k_0} is dense in $E\cdot[\alpha, \beta]$. For x in $E_{k_0}\cdot[\alpha, \beta]$, (2) holds for $n\geqq k_0$. For x in $(E-E_{k_0})\cdot[\alpha, \beta]$, (2) holds for $n\geqq k_0$, by continuity. Thus (2) holds uniformly for x in $E\cdot[\alpha, \beta]$, $n\geqq k_0$.

* This fact can be obtained as a special case of a general theorem in the theory of operations, which is established by similar reasoning; see S. Banach, op. cit., p. 19 (Theorem 11).

Let (x_0, y_0) be a complementary interval of the nowhere dense set $E \cdot [\alpha, \beta]$. Then the Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

converges to $f(x)$ for $x_0 \leqq x < y_0$. This follows at once from the facts that $f^{(k)}(x)$ is continuous on $x_0 \leqq x < y_0$ for $k = 0, 1, 2, \cdots$, and that $y_0 - x_0 < \delta/4 < \delta$.

Define an auxiliary function $\phi(x) = (\beta_1 - \alpha)/(\beta_1 - x)$, where $\delta/4 > \beta_1 - \beta > \beta - \alpha > 0$. The function $\phi(x)$ is analytic on $[\alpha, \beta]$ and is represented over the whole of $[\alpha, \beta]$ by its Taylor development about any point of $[\alpha, \beta]$. We have

$$\phi^{(k)}(x) = \frac{(\beta_1 - \alpha) \cdot k!}{(\beta_1 - x)^{k+1}} \geqq \frac{(\beta_1 - \alpha) \cdot k!}{(\beta_1 - \alpha)^{k+1}} \geqq k! \lambda^k,$$
$$(k = 0, 1, 2, \cdots ; \alpha \leqq x \leqq \beta).$$

Now form $\psi(x) = \phi(x) - f(x)$. The function $\psi(x)$ is represented by its Taylor development about x_0 for $x_0 \leqq x < y_0$; for $n \geqq k_0$, $\psi^{(n)}(x_0) \geqq 0$ and $\psi^{(n)}(y_0) \geqq 0$ by (2). Hence for $n \geqq k_0$, $\psi^{(n)}(x) \geqq 0$ for $x_0 \leqq x \leqq y_0$, since we may differentiate a power series termwise any number of times in the interior of its interval of convergence, so that $\psi^{(n)}(x)$ is represented over $x_0 \leqq x < y_0$ by a series of non-negative terms, for $n \geqq k_0$. This reasoning applies to any complementary interval of $E \cdot [\alpha, \beta]$, with the same function $\psi(x)$. Hence $\psi^{(n)}(x) \geqq 0$ for $\alpha \leqq x \leqq \beta$, $n \geqq k_0$. By a well known theorem of S. Bernstein, $\psi^{(k_0)}(x)$ is analytic for $\alpha \leqq x \leqq \beta$, and consequently $\psi(x)$ is analytic in the same interval. But then $f(x) = \phi(x) - \psi(x)$ is analytic in $[\alpha, \beta]$, contrary to the hypothesis that H was not an empty set. Hence H is an empty set, and the theorem is proved.

HARVARD UNIVERSITY

A SET OF DEFINING RELATIONS FOR THE SIMPLE GROUP OF ORDER 1092*

BY ABRAHAM SINKOV

1. *Introduction.* This paper is essentially a continuation of a previous paper, by H. R. Brahana, entitled *Certain perfect groups generated by two operators of orders two and three,*† the major portion of which is devoted to the relations $S^3 = T^2 = (ST)^7 = 1$. The groups defined by these relations are all perfect, that is, they coincide with their commutator subgroups. Consequently, the order of the commutator of S and T is included as an additional defining relation. The purpose of this note is to study the case when this commutator is of order 7 and to show that the relations

$$S^3 = T^2 = (ST)^7 = (S^{-1}T^{-1}ST)^7 = 1$$

completely define the simple group of order 1092. This result is then made use of to show that the non-alternating simple group of order 20,160 can not be generated by two operators of orders 2 and 3.

2. *Equivalent Generators.* It is shown in Brahana's paper that the generators S and T may be replaced by a pair of equivalent generators Q and R, in the sense that $\{S, T\}$ is the same as $\{Q, R\}$. The two sets of generators are connected by the relations

$$S = R^2Q, \qquad T = Q^{-1}R^4.$$

In terms of Q and R, the defining relations

$$S^3 = T^2 = (ST)^7 = (S^{-1}T^{-1}ST)^7 = 1$$

become

$$A: \qquad R^7 = Q^7 = 1, \qquad Q^{-1}R^4 = R^2Q, \qquad R^2QR^{-1} = Q^{-1}RQ.$$

The method of determining the order of the group G consists of an enumeration of the co-sets of G as regards the cyclic group

* Presented to the Society, February 23, 1935.

† American Journal of Mathematics, vol. 50 (1928), p. 315

generated by R; the proof of the completeness of this enumeration will follow from the representation of R as a permutation on the symbols designating the various co-sets. This permutation shows how the co-sets are permuted under multiplication on the right by R.

The notation is the same as that used by Brahana. Thus, the co-set $e=Q^2R$ is made up of the seven operators $R^\alpha Q^2R$, $(\alpha=0, 1, \cdots, 6)$, and e_i is the co-set $R^\alpha Q^2RQ^i$, $(i=1, 2, \cdots, 6)$.

3. *Definition of the Simple Group of Order* 1092. It will be convenient to list a series of relations, obtainable from A, which greatly simplify the reduction to the final number of co-sets. The verification of these relations is, in some cases, lengthy but it is not deemed necessary to reproduce the manipulation here.

1. $Q^\alpha R^2 = Q^{\alpha-1}R^4Q^{-1}$.
2. $Q^{-1}R^{4+\alpha} = R^3QR^\alpha$.
3. $QR^2 = RQRQ^{-1}$.
4. $QR^6 = RQR^4Q$.
5. $Q^4R = R^6Q^3R^6Q^3$.
6. $Q^4R^2 = RQ^3R^5Q^2$.
7. $Q^4R^4 = R^3Q^2R^4Q^2$.
8. $Q^4R^6 = R^5Q^2R^5Q^2$.
9. $Q^5R^2 = R^5Q^3RQ^3$.
10. $Q^5R^4 = R^4QR^4Q^2$.
11. $Q^5R^5 = R^2Q^2R^2Q^2$.
12. $Q^5R^6 = R^3Q^2RQ^6$.

With the aid of these relations it follows that every combination of Q and R is reducible to an operator in some one of the co-sets listed below.

$a = 1$, $i = Q^2R^6$, $q = QRQ^2R^4$,
$b = QR$, $j = Q^3R$, $r = QRQ^3R^4$,
$c = QR^4$, $k = Q^3R^2$, $s = QR^4Q^3R^2$,
$d = QR^5$, $l = Q^3R^4$, $t = QR^4Q^3R^6$,
$e = Q^2R$, $m = Q^3R^5$, $u = QR^4Q^5R$,
$f = Q^2R^2$, $n = Q^3R^6$, $v = QR^5Q^3R^4$,
$g = Q^2R^4$, $o = Q^4R^5$, $w = Q^2R^4Q^3R^2$,
$h = Q^2R^5$, $p = Q^5R$, $x = QRQ^2R^4Q^3R^2$.

Each of the above letters, with the exception of v and x, takes on every subscript from 1 to 6. The two co-sets $R^\alpha QR^5Q^3R^4$ and $R^\alpha QRQ^2R^4Q^3R^2$ are invariant under multiplication on the right by Q, so that neither v nor x may take any subscripts at all.

The representation of Q and R as permutations of these 156 symbols is as follows:

$$Q = (aa_1a_2a_3a_4a_5a_6)(bb_1b_2b_3b_4b_5b_6)\cdots$$
$$\cdots(uu_1u_2u_3u_4u_5u_6)(v)(ww_1w_2w_3w_4w_5w_6)(x).$$
$$R = (a)(g_4)(a_1bb_6a_6cdc_1)(a_2efc_6ghi)$$
$$(a_2jkg_6lmn)(a_4n_2m_2l_6g_2oh_2)(a_5pj_2g_1c_2f_2e_6)$$
$$(b_1f_1b_5d_6i_6p_1d_1)(b_2o_6e_5i_5qn_4j_4)(b_3m_6h_2q_6rj_1m_3)$$
$$(b_4h_6n_6r_6p_2e_1k_1)(c_3h_4sp_6l_5k_5t)(c_4i_3q_4l_4l_2r_2p_4)$$
$$(c_5uo_2l_1i_1s_2j_6)(d_2k_2o_5d_5r_5e_3q_1)(d_3d_4l_3r_4vq_2u_4)$$
$$(e_2u_3o_4n_5k_3l_4e_4)(f_3s_1f_6j_2l_6u_1h_1)(f_4r_1j_5l_5m_5i_4o_2)$$
$$(f_5k_4p_3m_4u_2h_5q_5)(g_3n_1wn_2g_5s_5s_4)(i_2p_5w_5u_6l_3t_1w_2)$$
$$(k_6o_1w_1s_6m_1s_3w_6)(q_3u_5xt_2r_3w_3w_4).$$

These two substitutions then define a transitive group on 156 symbols. Since G contains an operator of order 7 and is perfect, the following theorem results.

THEOREM 1. *The relations*

$$S^3 = T^2 = (ST)^7 = (S^{-1}T^{-1}ST)^7 = 1$$

completely define the simple group of order 1092.

Since the subgroup $\{a, g_4\}$ is of order 14, it may be used as the basis of a transitive representation of G readily obtainable from the preceding representation and involving only 18 letters.

Combining this result with the definition given by Brahana for the same group, we see that two generators of the simple group of order 1092, whose orders are 2 and 3 and whose product is of order 7, may have a commutator of order either 6 or 7.

4. *Generation by Two Operators.* Consider any group G generated by two operators of orders 2 and 3:

$$S^3 = T^2 = 1.$$

The commutator subgroup* H of G is generated by

* G. A. Miller, *On the commutator groups*, this Bulletin, vol. 4 (1897), p. 136.

$$\sigma_1 = S^{-1}TST, \qquad \sigma_3 = TS^{-1}TS,$$
$$\sigma_2 = TSTS^{-1}, \qquad \sigma_4 = STS^{-1}T.$$

But $\sigma_1=\sigma_3^{-1}$, $\sigma_2=\sigma_4^{-1}$. Therefore $H\equiv\{\sigma_1, \sigma_2\}$. Now, σ_1 and σ_2 are of the same order. Further

$$\sigma_1\cdot\sigma_2 = S^{-1}TS^{-1}TS^{-1} = S^{-1}(TS^{-1}TS)S,$$

and is of the same order as σ_1. Therefore H is generated by two operators of order n, whose product is of order n.

Returning to the groups defined by

$$S^2 = T^2 = (ST)^7 = 1$$

which are perfect, we see that they may also be generated by the two operators σ_1 and σ_2.

THEOREM 2. *The simple group of order* 1092 *may be generated by two operators of order* 6 *whose product is of order* 6, *or by two operators of order* 7 *whose product is of order* 7.

5. *Non-alternating Simple Group of Order* 20,160. Adding the results of this paper to those already obtained by Brahana, the groups defined by the relations

$$S^2 = T^2 = (ST)^7 = 1$$

are known for every order of $S^{-1}T^{-1}ST$ up to and including 7. These results may be applied to the non-alternating simple group of order 20,160 as follows. The operators contained within this group* must be of order 2, 3, 4, 5, or 7. If then, the group may be generated by two operators of orders 2 and 3 the product of these generators must be of order 7. But the order of their commutator could not exceed 7, and a contradiction results.

THEOREM 3. *It is not possible for two operators of orders* 2 *and* 3 *to generate the non-alternating simple group of order* 20,160.

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* L. E. Dickson, *Linear Groups with an Exposition of the Galois Field Theory*, p. 259.

ON THE NOTION OF REGULARITY OF METHODS OF SUMMATION OF INFINITE SERIES

BY J. D. TAMARKIN*

Let $\mathfrak{A}=(a_{mn})$ be the matrix of a method of summation which consists in replacing a given sequence $x=(x_1, x_2, \cdots)$ by its *transform*

$$(1) \quad y=(y_1, y_2, \cdots), \quad y_m=\sum_{n=1}^{\infty} a_{mn}x_n, \qquad (m=1, 2, \cdots),$$

and defining the generalized limit of x_n as $\lim_{m\to\infty} y_m$, provided this limit exists.

The method $\mathfrak{A}$ is called regular if every convergent sequence $x=\{x_n\}$ is transformed into a convergent sequence $y=\{y_m\}$ with the same limit. Necessary and sufficient conditions for regularity of $\mathfrak{A}$ are too well known to be restated here. An essential point in the whole theory is the assumption that the sequence $y\equiv y(x)$ is determined by formula (1) for *each* convergent sequence x. A (quite trivial) gain in generality may be achieved by demanding that not all the terms y_m of the sequence $\{y_m\}$ have to be considered but only those for which $m\geqq m_0$, where m_0 is a fixed integer. Even this requirement is not at all necessary, however, for the possibility of evaluating $\lim_{m\to\infty} y_m$, for which we have to know only *almost all* terms of the sequence $\{y_m\}$, that is all y_m, $m\geqq m'$, where m' need not be fixed, but on the contrary, may depend on the sequence x.

Thus we are naturally led to the following apparently less restrictive definition of regularity of the method of summation $\mathfrak{A}$.

The method of summation $\mathfrak{A}$ *is regular if* (i) *to every convergent sequence* $x=\{x_n\}$ *there corresponds an integer* $m'(x)$ *such that* y_m *as given by* (1) *exists for* $m\geqq m'(x)$, *and* (ii) *for a fixed* x,

$$\lim_{m\to\infty} y_m = \lim_{n\to\infty} x_n.$$

It turns out, however, that the modified definition of regularity

* The result of this note answers a question raised by Dr. H. Lewy.

actually is not more general than the classical one. Indeed we intend to show that if $\mathfrak{A}$ satisfies condition (i) above, then there exists a fixed number m_0 which does not depend on x, such that all the series

$$y_m = \sum_{n=1}^{\infty} a_{mn}x_n, \qquad (m \geqq m_0)$$

converge.

Assume that the number in question does not exist. Then we can find a sequence of integers $\{m_1, m_2, \cdots\}$, $m_p \to \infty$ and a set of sequences $x^{(p)} = \{x_n^{(p)}\}$, such that none of the series

$$\sum_{n=1}^{\infty} a_{mn}x_n^{(p)}, \qquad (m = m_p;\ p = 1, 2, \cdots),$$

converges. Now introduce the space (c) of all convergent sequences x. This space is a linear metric complete space if we introduce the usual definitions of the sum of two elements and of the product of an element by a scalar, and if we define the norm of x by

$$\|x\| = \sup_n |x_n|.$$

Consider the double array of linear functionals on (c) defined by

$$U_{pq}(x) = \sum_{n=1}^{q} a_{mn}x_n, \qquad (m = m_p;\ p, q = 1, 2, \cdots).$$

Our assumption is that none of the limits

$$\lim_{q\to\infty} U_{pq}(x^{(p)}), \qquad (p = 1, 2, \cdots),$$

exists. Then, by an important theorem of Banach,* the set H of elements x for which none of the limits

$$\lim_{q\to\infty} U_{pq}(x), \qquad (p = 1, 2, \cdots),$$

exists is of the second category, hence not empty. This means that there exists a convergent sequence $x = \{x_n\}$ such that no series

* S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, pp. 24–25, Theorem 4.

$$\sum_{n=1}^{\infty} a_{mn}x_n, \qquad (m = m_1, m_2, \cdots),$$

converges. This however cannot occur if $\mathfrak{A}$ is regular in the sense of the definition above, according to which all series

$$\sum_{n=1}^{\infty} a_{mn}x_n, \qquad (m \geqq m'(x)),$$

must converge.

The existence of a fixed m_0 is thus established, and at the same time it is shown that our modified definition of regularity of $\mathfrak{A}$ is equivalent to the classical one. It is clear that analogous considerations can be applied when instead of summation of series we deal with summation of integrals.

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AN EXTENSION TO POLYGAMMA FUNCTIONS OF A THEOREM OF GAUSS*

BY H. T. DAVIS

1. *Introduction.* By the polygamma functions we mean the derivatives of $\log \Gamma(x)$, that is, $\Psi(x) = \Gamma'(x)/\Gamma(x)$, $\Psi'(x), \cdots, \Psi^{(n)}(x)$.† These functions satisfy the difference equations:

$$\Psi^{(n)}(x+1) - \Psi^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}},$$

(1)

$$\Psi^{(n)}(1-x) + (-1)^{n+1}\Psi^{(n)}(x) = A_n(x), \quad A_n(x) = \frac{d^n}{dx^n}(\pi \operatorname{ctn} \pi x),$$

subject to the boundary condition,

$$\Psi^{(n)}(1) = (-1)^{n+1} n! S_{n+1},$$

where we employ the abbreviation

$$S_m = 1 + 1/2^m + 1/3^m + \cdots.$$

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* Presented to the Society, December 27, 1934.

† The name polygamma is suggested by the paper, *Tables of the Digamma and Trigamma Functions*, by Eleanor Pairman, Tracts for Computers, No. 1, 1919.

One of the most striking identities satisfied by these functions is the following:

$$\Psi^{(m)}(nx) = (1/n^{m+1})\left[\Psi^{(m)}(x) + \Psi^{(m)}(x + 1/n) + \cdots + \Psi^{(m)}(x + 1 - 1/n)\right], \tag{2}$$

which holds for $m>0$. When $m=0$ the right member must be augmented by $\log n$.

In the construction of his celebrated tables of $\Psi(x)$ to 18 decimal places, Gauss* did not avail himself of the asymptotic expansion of this function, but employed rather a singularly elegant theorem by means of which values of $\Psi(x)$ for rational values of the argument could be explicitly determined. Although most of the theorems and identities associated with the polygamma functions can be derived by the simple device of differentiating the corresponding identities associated with $\Psi(x)$, this method is not possible for the arithmetical theorem of Gauss.

It is the purpose of this note, therefore, to develop the analogous expansions for $\Psi^{(n)}(p/q)$, where p and q are integers, $p<q$.

2. *The Theorem.* We shall prove the following theorem.

If p and q are integers, $p<q$, and if n is greater than 0, then the following formulas hold:

$$\Psi^{(n)}(p/q) + \Psi^{(n)}(1 - p/q) = (-1)^{n+1}2\cdot n!\, q^n \sum_{r=1}^{q-1} \cos(2\pi rp/q) L^{(n+1)}(2\pi r/q) + 2q^n\Psi^{(n)}(1),$$

$$\Psi^{(n)}(p/q) - \Psi^{(n)}(1 - p/q) = (-1)^{n+1}2\cdot n!\, q^n \sum_{r=1}^{q-1} \sin(2\pi rp/q) M^{(n+1)}(2\pi r/q),$$

where we employ the abbreviations

$$L^{(m)}(x) = \sum_{n=1}^{\infty} \cos nx/n^m, \qquad M^{(m)}(x) = \sum_{n=1}^{\infty} \sin nx/n^m.$$

PROOF. If in the definitive expansion

* *Disquisitiones Generales Circa Seriem Infinitam*, Werke, vol. 3, pp. 125–162.

$$\Psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} 1/(x+k)^{n+1}, \qquad (n > 0),$$

we replace x by r/q, multiply each $\Psi^{(n)}(r/q)$ by $\cos(2m\pi r/q)$, and form the sum $\sum_{r=1}^{q} \cos(2m\pi r/q)\Psi^{(n)}(r/q)$, we shall have

$$\sum_{r=1}^{q} \cos(2m\pi r/q)\Psi^{(n)}(r/q) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \sum_{r=1}^{q} \cos(2m\pi r/q) q^{n+1}/(r+qk)^{n+1}.$$

But if we note that $\cos[2m\pi(sq+r)/q] = \cos(2m\pi r/q)$, then this sum takes the simple form

$$\sum_{r=1}^{q} \cos(2m\pi r/q)\Psi^{(n)}(r/q) = (-1)^{n+1} n! q^{n+1} \sum_{r=1}^{\infty} \cos(2m\pi r/q)/r^{n+1} = (-1)^{n+1} n! q^{n+1} L^{(n+1)}(2m\pi/q).$$

If now we let m be successively $1, 2, 3, \cdots, q-1$ and note that $\cos(2m\pi)\Psi^{(n)}(1) = (-1)^{n+1} n! S_{n+1}$, we obtain the following system of equations:

$$\sum_{r=1}^{q-1} \cos(2m\pi r/q)\Psi^{(n)}(r/q) = (-1)^{n+1} n! q^{n+1} L^{(n+1)}(2m\pi/q) + (-1)^{n} n! S_{n+1}, \qquad (m = 1, 2, \cdots, q-1).$$

To this system we then add the equation

$$\sum_{r=1}^{q-1} \cos(2\pi r)\Psi^{(n)}(r/q) = (q^{n+1} - 1)(-1)^{n+1} n! S_{n+1},$$

which is recognized as equation (2) in which the members of the right-hand side have been successively multiplied by $\cos 2\pi$, $\cos 4\pi$, etc.

We now multiply each equation of the system successively by $\cos(2\pi p/q)$, $\cos(4\pi p/q)$, $\cos(6\pi p/q)$, etc., and add, taking note of the well known identity

$$I_{pr}(q) = \sum_{n=1}^{q} \cos(2\pi np/q) \cos(2\pi nr/q) = \begin{cases} 0 \text{ if neither } p-r \text{ nor } p+r \text{ is divisible by } q, \\ q/2 \text{ if either } p-r \text{ or } p+r \text{ but not both is divisible by } q, \\ q \text{ if both } p-r \text{ and } p+r \text{ are divisible by } q. \end{cases}$$

Hence when q is odd, we obtain the identity

$$\Psi^{(n)}(r/q) + \Psi^{(n)}(1 - r/q)$$

$$= (-1)^{n+1} n! \sum_{m=1}^{q-1} \cos(2\pi mr/q) q^n \cdot L^{(n+1)}(2\pi m/q) \tag{3}$$

$$+ 2(-1)^{n+1} q^n n! S_{n+1}.$$

But when q is even, there is no difference in argument. In that case there will be a middle term $\Psi^{(n)}(r/q) = \Psi^{(n)}(1-r/q) = \Psi^{(n)}(1/2)$. But since $I = q$, we shall have $(1/2)q\Psi^{(n)}(1/2) + (1/2)q\Psi^{(n)}(1/2) = (1/2)q\Psi^{(n)}(r/q) + (1/2)q\Psi^{(n)}(1-r/q)$, which is the same as the result obtained when n was odd.

In order to obtain the second identity stated in the theorem we now repeat the argument just given, merely replacing cosines by sines and $L^{(n)}(x)$ by $M^{(n)}(x)$ and noting the well known identity,

$$J_{pr}(q) = \sum_{n=1}^{q} \sin(2\pi np/q) \sin(2\pi nr/q)$$

$$= \begin{cases} 0 \text{ if neither } p-r \text{ nor } p+r \text{ is divisible by } q, \text{ or if both} \\ \quad \text{are divisible by } q, \\ q/2 \text{ if } p-r \text{ but not } p+r \text{ is divisible by } q, \\ -q/2 \text{ if } p+r \text{ but not } p-r \text{ is divisible by } q. \end{cases}$$

We then obtain

$$\Psi^{(n)}(r/q) - \Psi^{(n)}(1 - r/q)$$

$$= (-1)^{n+1} n! q^n \sum_{m=1}^{q-1} \sin(2\pi mr/q) M^{(n+1)}(2\pi m/q).$$

From these identities and known values of $L^{(n)}(x)$ and $M^{(n)}(x)$ it is possible to make explicit determinations of $\Psi^{(n)}(x)$ for rational values of the argument. We first note the formulas,*

$$\int_0^x L^{(n)}(t)dt = M^{(n+1)}(x), \quad \int_0^x M^{(n)}(t)dt = -L^{(n+1)}(x) + S_{n+1};$$

$$L^{(1)}(x) = -\log\left(2 \sin \frac{x}{2}\right), \quad M^{(1)}(x) = \frac{(\pi - x)}{2},$$

$$(0 < x < 2\pi);$$

* See T. J. I'a. Bromwich, *Infinite Series*, 2d ed., 1926, pp. 359–364.

$$L^{(2)}(x) = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12},$$

$$M^{(3)}(x) = \frac{1}{12}\{(x-\pi)^3 - \pi^2 x + \pi^3\}, \qquad (0 \leqq x \leqq 2\pi);$$

$$L^{(4)}(x) = \frac{1}{48}\left\{2\pi^2(x-\pi)^2 - (x-\pi)^4 - \frac{7\pi^4}{15}\right\}, \qquad (0 \leqq x \leqq 2\pi);$$

$$M^{(5)}(x) = \frac{1}{720}\{10\pi^2(x-\pi)^3 - 3(x-\pi)^5 - 7\pi^4 x + 7\pi^5\}, \qquad (0 \leqq x \leqq 2\pi).$$

In the general case $L^{(m)}(x)$ for even exponents and $M^{(m)}(x)$ for odd exponents can be expressed in terms of Bernoulli polynomials as follows.

$$L^{(2n)}(x) = (-1)^{n+1}2^{2n-1}\pi^{2n}B_{2n}(x/2\pi)/(2n)!,$$
$$M^{(2n+1)}(x) = (-1)^{n+1}2^{2n}\pi^{2n+1}B_{2n+1}(x/2\pi)/(2n+1)!.$$

Applying the theorem to the cases $q=2$, 3, and 4, we are able to derive the following values:

$$\Psi^{(n)}(1/2) = (-1)^{n+1}n!(2^{n+1}-1)S_{n+1},$$
$$\Psi^{(n)}(1/4) = (1/2)(-1)^{n+1}n!4^{n+1}[T_{n+1} + (1 - 1/2^{n+1})S_{n+1}],$$
$$\Psi^{(n)}(3/4) = (1/2)(-1)^{n+1}n!4^{n+1}[-T_{n+1} + (1 - 1/2^{n+1})S_{n+1}],$$
$$\Psi^{(n)}(1/3) = (1/2)(-1)^{n+1}n![3^{n+1}\sigma_{n+1} + (3^{n+1}-1)S_{n+1}],$$
$$\Psi^{(n)}(2/3) = (1/2)(-1)^{n+1}n![-3^{n+1}\sigma_{n+1} + (3^{n+1}-1)S_{n+1}],$$

where we employ the abbreviations

$$S_r = \sum_{m=1}^{\infty} 1/m^r, \qquad T_r = \sum_{m=1}^{\infty} (-1)^{m+1}/(2m-1)^r,$$
$$\sigma_r = 1 - 1/2^r + 1/4^r - 1/5^r + 1/7^r - 1/8^r + 1/10^r - 1/11^r + \cdots.$$

INDIANA UNIVERSITY

IRREDUCIBILITY OF POLYNOMIALS OF DEGREE n WHICH ASSUME THE SAME VALUE n TIMES*

BY LOUIS WEISNER

1. *Introduction.* A polynomial $F(x)$ of degree n, with integral coefficients, which assumes the same value k for n distinct integral values of x has the form

$$F(x) = a_0(x - a_1)(x - a_2) \cdots (x - a_n) + k, \qquad (a_0 \neq 0),$$

where the a's denote integers, and $a_1, a_2, \cdots, a_n$ are distinct. The irreducibility of polynomials of this type in the field of rational numbers has been discussed by several writers for the particular cases† $|k| = 1$, $|k| =$ prime.

The present paper is concerned with the irreducibility of $F(x)$ for the case in which k is any integer $\neq 0$. It is obvious that even when the a's are fixed, an infinitude of choices of k exists for which $F(x)$ is reducible. What is not obvious is that when k and n are fixed, *only a finite number of non-equivalent reducible polynomials of the form* $F(x)$ *exist.* Two polynomials $F(x)$ and $G(x)$, with integral coefficients, are regarded as *equivalent* if an integer h exists such that $F(x) = \pm G(\pm x + h)$. Moreover, if only k is fixed, but n is sufficiently large, *every* polynomial of the form of $F(x)$ is irreducible.

2. *Isolation of the Roots of* $f(x)$. The polynomial $F(x)$ of §1 is evidently equivalent to the polynomial

$$f(x) = ax(x - t_1) \cdots (x - t_{n-1}) \pm k,$$

where $a, k, t_1, \cdots, t_{n-1}$ are positive integers, and the t's are distinct. We shall confine our attention to $f(x)$ and assume that $n \geqq 2$. We shall denote by x_0 a root of $f(x)$ whose absolute value is a minimum, and the other roots by $x_1, \cdots, x_{n-1}$. Taking the ratio of the coefficient of x to the constant term in each of the last two members of

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* Presented to the Society, September 5, 1934.

† For literature, see Dorwart and Ore, Annals of Mathematics, vol. 34 (1933), p. 81; A. Brauer, Jahresbericht der Deutscher Mathematiker Vereinigung, vol. 43 (1933), p. 124.

$$(1) \qquad f(x) = ax\prod_{i=1}^{n-1}(x - t_i) \pm k = a\prod_{i=0}^{n-1}(x - x_i),$$

we have

$$\frac{at_1 \cdots t_{n-1}}{k} = \left| \frac{1}{x_0} + \cdots + \frac{1}{x_{n-1}} \right|.$$

Hence

$$(2) \qquad |x_0| \leqq \frac{nk}{at_1 \cdots t_{n-1}}.$$

In the same way we infer from

$$f(x + t_j) = ax(x + t_j)\prod_{i=1, i\neq j}^{n-1}(x + t_j - t_i) \pm k$$

$$= a\prod_{i=0}^{n-1}(x + t_j - x_i),$$

that, to each index $j \geqq 1$, there corresponds an index p such that

$$(3) \qquad |t_j - x_p| \leqq \frac{nk}{at_j\prod_{i=1, i\neq j}^{n-1}|t_j - t_i|}, \qquad (0 \leqq p \leqq n-1).$$

THEOREM 1. *If the inequalities*

$$(4) \qquad \begin{aligned} 2nk &< at_1 \cdots t_{n-1}, \\ 2nk &< at_j\prod_{i=1, i\neq j}^{n-1}|t_j - t_i|, \quad (j = 1, \cdots, n-1), \end{aligned}$$

are satisfied, the roots of $f(x)$ are all real and lie within the intervals

$$\left[-\frac{1}{2}, +\frac{1}{2}\right], \quad \left[t_j - \frac{1}{2}, t_j + \frac{1}{2}\right], \quad (j = 1, \cdots, n-1).$$

From (2), (3) and (4) we have

$$(5) \qquad |x_0| < \frac{1}{2}, \quad |t_j - x_p| < \frac{1}{2}, \quad (j = 1, \cdots, n-1).$$

These inequalities show that each of the n circles

$$(6) \qquad |x| = \frac{1}{2}, \quad |x - t_j| = \frac{1}{2}, \quad (j = 1, \cdots, n-1).$$

contains a root of $f(x)$. As $t_1, \cdots, t_{n-1}$ are distinct positive integers, no two of these circles intersect. It follows that each of the circles (6) contains one and only one root of $f(x)$. As a circle with center on the axis of reals which contains one of two conjugate imaginary numbers contains the other, while each of the circles (6) contains only one root of $f(x)$, the roots of $f(x)$ are real and lie within the stated intervals.* We shall choose our notation so that

$$(7) \qquad |t_i - x_i| < \frac{1}{2}, \quad (i = 1, \cdots, n-1).$$

3. *Irreducibility of $f(x)$.* It is convenient to define $\lambda = \lambda(n)$ by

$$(8) \qquad \begin{aligned} &\lambda(2) = 1, \quad \lambda(3) = 4, \quad \lambda(4) = 6, \quad \lambda(5) = 3, \quad \lambda(6) = 1, \\ &\lambda(n) = 0 \text{ if } n \geqq 7. \end{aligned}$$

THEOREM 2. *The polynomial $f(x)$ is irreducible if at least one of the n inequalities*

$$(9) \qquad a > 2^n k^2 + 1, \quad t_i > (3 + \lambda)k, \quad (i = 1, \cdots, n-1),$$

is satisfied.

With the aid of (8) and the fact that the t's are distinct positive integers, it is readily proved that each of the inequalities (9) implies all of the inequalities (4). The roots of $f(x)$ are therefore isolated as described by Theorem 1.

Suppose that $f(x)$ is reducible:

$$(10) \qquad f(x) = B(x)C(x) = \sum_{v=0}^{r} b_v x^{r-v} \cdot \sum_{v=0}^{s} c_v x^{s-v}, \qquad (b_0 c_0 \neq 0),$$

$(1 \leqq r \leqq n-1; 1 \leqq s \leqq n-1; r+s=n)$, the b's and c's being integers. Let $B(x)$ be that factor which has x_0 as a root; and let $x_1, \cdots, x_s$ be the roots of $C(x)$, so that

$$(11) \qquad C(x) = c_0 \prod_{i=1}^{s} (x - x_i) = \sum_{v=0}^{s} c_v x^{s-v}.$$

* The referee has called my attention to an alternative proof which consists in showing that $f(1/2)$ and $f(-1/2)$ have opposite signs, and that $f(t_i - 1/2)$ and $f(t_i + 1/2)$ have opposite signs.

By Theorem 1, the roots of $C(x)$ are $>1/2$, and at most one of them is <1.

As $b_s c_s = \pm k$,

$$k \geqq \left| \frac{c_s}{c_0} \right| = x_1 \cdots x_s. \tag{12}$$

Hence

$$|x_j| \leqq 2k, \qquad (j = 1, \cdots, s). \tag{13}$$

Substituting $x = t_i$ in (10), we have

$$B(t_i)C(t_i) = f(t_i) = \pm k, \qquad (i = 1, \cdots, n-1).$$

Hence

$$|c_0| \prod_{j=1}^{s} |t_i - x_j| = |C(t_i)| \leqq k.$$

As $|c_0| \geqq 1$, an index j exists such that

$$|t_i - x_j| \leqq k, \qquad (1 \leqq j \leqq s).$$

It follows from (13) that

$$t_i \leqq 3k, \quad (i = 1, \cdots, n-1). \tag{14}$$

Multiplying the equations

$$a x_j \prod_{i=1}^{n-1} |x_j - t_i| = k, \qquad (j = 1, \cdots, s),$$

obtained by substituting $x = x_j$ in (1), we have by (11),

$$a^s |c_s| \prod_{i=1}^{n-1} |C(t_i)| = k^s |c_0^n|. \tag{15}$$

From the nature of the roots of $C(x)$ and (12), we have $|c_s| \geqq |c_0|/2$. As c_s is a divisor of k, $|c_0| \leqq 2k$. As $C(t_i)$ is an integer $\neq 0$, it follows from (15) that $a^s \leqq 2^n k^{n+s-1}$. If s has its maximum value $n-1$, this inequality becomes $a^{n-1} \leqq 2^n k^{2(n-1)}$, whence $a \leqq 2^n k^2$. If $s < n-1$, we have, with the notation (7),

$$|t_i - x_j| > \frac{1}{2}, \qquad (i = s+1, \cdots, n-1).$$

(The right member may be replaced by 1 for all but one x_j.) Hence $|C(t_i)| > |c_0|/2$, and

$$\prod_{i=s+1}^{n-1} |C(t_i)| > \frac{|c_0^{n-s-1}|}{2^{n-s-1}}.$$

It follows from (15) that

$$a^s \leqq 2^{n-s}k^s |c_0|^s \leqq 2^n k^{2s},$$

whence $a \leqq 2^n k^2$. As this inequality, and (14), contradict (9), we conclude that $f(x)$ is irreducible.

The example

$$b^2x(x-1)(x-3)(x-4) - 3b - 1$$
$$= (bx^2 - 4bx + 3b + 1)(bx^2 - 4bx - 1),$$

in which $a = b^2$, $\pm k = 3b+1$, shows that the first of the inequalities (9) cannot be replaced by one which is linear in k.

While the inequalities (9) can undoubtedly be weakened by further analysis, without affecting the irreducibility of $f(x)$, they suffice to establish the following general theorems.

THEOREM 3. *Only a finite number of non-equivalent reducible polynomials of degree n exist which assume a given integral value* $\neq 0$ *for n different integral values of the variable.*

For, if n and k are fixed positive integers, only a finite number of sets of positive integers $a, t_1, \cdots, t_{n-1}$ exist which violate all the inequalities (9).

THEOREM 4. *If k is a fixed integer* $\neq 0$, *and n is sufficiently large, every polynomial of degree n which assumes the value k for n distinct integral values of its argument is irreducible.*

At least one of the integers $t_1, \cdots, t_{n-1}$ is $\geqq n-1$. Hence if $n \geqq 7$ and $n > 3k+1$, at least one of the inequalities

$$t_i > (3 + \lambda)k, \qquad (i = 1, \cdots, n-1),$$

is satisfied, and $f(x)$ is irreducible.

HUNTER COLLEGE OF THE CITY OF NEW YORK

NOTE ON THE EQUATION OF HEAT CONDUCTION*

BY A. E. HEINS

1. *Introduction.* In recent years, operational methods in mathematics have been a subject of much discussion. Heaviside, in his papers, by somewhat artificial methods, succeeded in solving a number of differential equations, especially those common to the electro-magnetic theory. The newer developments in operational calculus make no attempt to follow Heaviside's methods. More recent literature shows that the Fourier integral (Jeffreys,† Bush‡) plays an important role in these methods. In a thesis of Levinson,§ it was demonstrated that the Fourier transform could be employed to even better advantage than the Fourier integral.

In this note, it is proposed to quote the Fourier transform theorem of several variables, and apply a particular form of it to the solution of the equation for the flow of heat in three dimensions.

2. *Fourier Transform of Several Variables.* Here we shall consider a function of k real variables $F(x_1, \cdots, x_k)$ in a closed domain $a_\lambda \leqq x_\lambda \leqq b_\lambda$, $(\lambda = 1, \cdots, k)$, capable of taking on complex values. Consider an integral with infinite limits, such as

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(x_1, \cdots, x_k) dx_1 \cdots dx_k.$$

Such an integral is said to be convergent if the limit of the integral

$$\lim_{A_\lambda \to \infty, B_\lambda \to \infty} \int_{-B_1}^{A_1} \cdots \int_{-B_k}^{A_k} F(x_1, \cdots, x_k) dx_1 \cdots dx_k, \qquad (\lambda = 1, \cdots, k),$$

* From a thesis presented for the degree Master of Science at the Massachusetts Institute of Technology, Oct. 30, 1934, under the title *Applications of the Fourier transform theorem.* Presented to the Society, December 28, 1934.

† H. Jeffreys, *Operational Methods in Mathematical Physics*, Cambridge Tract No. 23, 1931.

‡ V. Bush, *Operational Circuit Analysis*, 1929.

§ N. Levinson, *Applications of the Fourier integral*, master's thesis, May, 1934, Massachusetts Institute of Technology, unpublished as yet.

is determined. If, moreover,

$$\int_{-\infty}^{\infty} |F(x_1, \cdots, x_k)|\, dx_\lambda < \infty, \qquad (\lambda = 1, \cdots, k),$$

then $F(x_1, \cdots, x_k)$ is called absolutely integrable.

Let there be given a function $F(x_1, \cdots, x_k)$ with the following properties:

1. It has only a finite number of finite discontinuities.
2. It is piecewise continuous.
3. It is absolutely integrable and bounded.

Then under these hypotheses the Fourier integral of the function $F(x_1, \cdots, x_k)$ exists and is given by the following expression:

$$(2\pi)^k F(x_1, \cdots, x_k)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(t_1, \cdots, t_k) e^{-i\sum_{\lambda=1}^{k} [u_\lambda (t_\lambda - x_\lambda)]} dt_1 du_1 \cdots dt_k du_k.$$

By rearranging this integral, the Fourier transform theorem is obtained. Denote the integral

$$\left(\frac{1}{2\pi}\right)^{k/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(x_1, \cdots, x_k) e^{-i\sum_{\lambda=1}^{k} t_\lambda x_\lambda} dx_1 \cdots dx_k$$

by $g(t_1, \cdots, t_k)$. Then we have

$$(2\pi)^{k/2} F(x_1, \cdots, x_k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(t_1, \cdots, t_k) e^{i\sum_{\lambda=1}^{k} t_\lambda x_\lambda} dt_1 \cdots dt_k.$$

That is, $g(t_1, \cdots, t_k)$ is the transform of $F(x_1, \cdots, x_k)$ and, reciprocally, $F(x_1, \cdots, x_k)$ is the transform of $g(t_1, \cdots, t_k)$. This dual character of the Fourier transform theorem plays an important role in the solution of certain partial differential equations.

In the particular problem treated here, we make use of a special form of the transform theorem, commonly called the Mellin transform. To bring the Fourier transform into this form, consider first a function $F_1(x_1, \cdots, x_k)$ given by

$$F_1(x_1, \cdots, x_k) = e^{-\sum_{\lambda=1}^{k} c_\lambda x_\lambda} F(x_1, \cdots, x_k), \qquad (c_\lambda > 0).$$

Then the Fourier integral for $F_1(x_1, \cdots, x_k)$ becomes

$$\left(\frac{1}{2\pi}\right)^k \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-i\sum_{\lambda=1}^{k}[u_\lambda(t_\lambda - x_\lambda) - ic_\lambda t_\lambda]} F_1(t_1, \cdots, t_k) dt_1 du_1 \cdots dt_k du_k.$$

Let $u_\lambda = v_\lambda + ic_\lambda$. Then $F_1(x_1, \cdots, x_k)$ is

$$\left(\frac{1}{2\pi}\right)^k \int_{-ic_1-\infty}^{-ic_1+\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i\sum_{\lambda=1}^{k}[(v_\lambda + ic_\lambda)x_\lambda]} \psi dt_1 dv_1 \cdots dt_k dv_k,$$

where ψ is

$$F_1(t_1, \cdots, t_k) e^{-\sum_{\lambda=1}^{k} c_\lambda t_\lambda} e^{-i\sum_{\lambda=1}^{k}(v_\lambda + ic_\lambda)t_\lambda},$$

or the value of $F(x_1, \cdots, x_k)$ is

$$\left(\frac{1}{2\pi}\right)^k \int_{-ic_1-\infty}^{-ic_1+\infty} \int_{-\infty}^{\infty} \cdots$$

$$\int_{-\infty}^{\infty} F(t_1, \cdots, t_k) e^{-i\sum_{\lambda=1}^{k} v_\lambda (t_\lambda - x_\lambda)} dt_1 dv_1 \cdots dt_k dv_k.$$

Now put $p_\lambda = iv_\lambda$ and we obtain for $F(x_1, \cdots, x_k)$ the value

$$\left(\frac{1}{2\pi i}\right)^k \int_{-i\infty+c_1}^{i\infty+c_1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(t_1, \cdots, t_k) e^{-\sum_{\lambda=1}^{k} p_\lambda (t_\lambda - x_\lambda)} dt_1 dp_1 \cdots dt_k dp_k.$$

If this integral is split into two parts as the Fourier integral was, the Mellin transform is obtained. That is,

$$g(p_1, \cdots, p_k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(t_1, \cdots, t_k) e^{-\sum_{\lambda=1}^{k} p_\lambda t_\lambda} dt_1 \cdots dt_k,$$

and

$$F(x_1, \cdots, x_k) = \left(\frac{1}{2\pi i}\right)^k \int_{-i\infty+c_1}^{i\infty+c_1} \cdots \int_{-i\infty+c_k}^{i\infty+c_k} g(p_1, \cdots, p_k) e^{\sum_{\lambda=1}^{k} p_\lambda x_\lambda} dp_1 \cdots dp_k.$$

In physical problems we require oftentimes no knowledge of the past state of the system. That is, we are only interested in everything which takes place when t (time) $\geqq 0$. It is for this reason that it is of convenience to arrange coordinates such that

$F(x_1, \cdots, x_k)=0$ for all $x_\lambda<0$. Then $g(p_1, \cdots, p_k)$ becomes under this restriction

$$\int_0^\infty \cdots \int_0^\infty F(t_1, \cdots, t_k)e^{-\sum_{\lambda=1}^k p_\lambda t_\lambda}dt_1 \cdots dt_k,$$

since the interval $(-\infty, 0)$ does not contribute anything to this transform.

There is just one more theorem which is of interest. Given $f_1(x_1, \cdots, x_k)$ and $f_2(x_1, \cdots, x_k)$ and their respective transforms $g_1(p_1, \cdots, p_k)$ and $g_2(p_1, \cdots, p_k)$, what is the transform of $g_1(p_1, \cdots, p_k)g_2(p_1, \cdots, p_k)$? This is answered by the faltung theorem. That is, if the function $f_1(x_1, \cdots, x_k)$ is the transform of $g_1(p_1, \cdots, p_k)$ and the function $f_2(x_1, \cdots, x_k)$ is the transform of $g_2(p_1, \cdots, p_k)$, then the transform of the function $g_1(p_1, \cdots, p_k)\, g_2(p_1, \cdots, p_k)$ is given by

$$f(x_1, \cdots, x_k) = \int_0^{x_1} \cdots \int_0^{x_k} f_1(y_1, \cdots, y_k) \cdot f_2(x_1 - y_1, \cdots, x_k - y_k)dy_1 \cdots dy_k.$$

We must remark that this theorem holds under the restriction that $f_2(x_1-y_1, \cdots, x_k-y_k)=0$ for $y_\lambda>x_\lambda$ and $f_1(y_1, \cdots, y_k)=0$ for $y_\lambda<0$.*

3. *Heat Flow in an Infinite Medium.* We propose to solve here the following linear parabolic differential equation

$$K\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\right)=\frac{\partial V}{\partial t}. \tag{1}$$

Multiply equation (1) by $e^{-p_1x-p_2y-p_3z}$ and integrate from 0 to ∞ over x, y, and z. Then we find by the transform theorem

$$\frac{\partial g}{\partial t}=K(p_1^2+p_2^2+p_3^2)g. \tag{2}$$

Thus we see that p_1, p_2, p_3 act as partial derivatives in the sense that p_1 acting on g is the transform of $\partial V/\partial x$. Similarly p_1^2

* See S. Bochner, *Vorlesungen über Fouriersche Integrale*, 1932, for a complete discussion of the Fourier transform.

acting on g is the transform of $\partial^2 V/\partial x^2$. Solving equation (2) for g we get

$$g = Ae^{kt(p_1^2+p_2^2+p_3^2)}, \tag{3}$$

where A is a function of p_1, p_2, p_3 to be determined.

For initial conditions we put the original temperature distribution equal to $f(x, y, z)$ when $t=0$. Moreover, without any lack of generality, we can limit the flow to the positive octant so that $V=0$ for $x<0$, $y<0$, $z<0$. Let $H(p_1, p_2, p_3)$ be the transform of $f(x, y, z)$. Then $A(p_1, p_2, p_3)=H(p_1, p_2, p_3)$ and

$$V = \left(\frac{1}{2\pi i}\right) \int_{-i\infty+c_1}^{i\infty+c_1} \int_{-i\infty+c_2}^{i\infty+c_2} \int_{-i\infty+c_3}^{i\infty+c_3} He^{k(p_1^2+p_2^2+p_3^2)t} \cdot e^{p_1x+p_2y+p_3z} dp_1 dp_2 dp_3. \tag{4}$$

To evaluate this expression, consider the transform of

$$e^{kt(p_1^2+p_2^2+p_3^2)}.$$

That is, we must evaluate

$$\left(\frac{1}{2\pi i}\right)^3 \int_{-i\infty+c_1}^{i\infty+c_1} e^{kt(p_1^2+p_1x/(kt))} dp_1 \int_{-i\infty+c_2}^{i\infty+c_2} e^{kt(p_2^2+p_2y/(kt))} dp_2 \cdot \int_{-i\infty+c_3}^{i\infty+c_3} e^{kt(p_3^2+p_3z/(kt))} dp_3.$$

This integral may be considered as the product of three integrals, for example,

$$\left(\frac{1}{2\pi i}\right) \int_{-i\infty+c_1}^{i\infty+c_1} e^{kt(p_1^2+p_1x/(kt))} dp_1. \tag{5}$$

This may be written as

$$\frac{e^{-x^2/(4kt)}}{2\pi i} \int_{-i\infty+c_1}^{i\infty+c_1} e^{kt(p_1+x/(2kt))^2} dp_1. \tag{6}$$

If we put $iu_1=(kt)^{1/2}(p_1+x/(2kt))$, then equation (5) becomes

$$\frac{e^{-x^2/(4kt)}}{2\pi(kt)^{1/2}} \int_{-\infty+iB}^{\infty+iB} e^{-u_1^2} du_1,$$

where the B's include variables which do not depend on our

integration variable. The integral in equation (6) is taken along a path parallel to the x axis. This contour may be deformed and the integral can be taken along the entire x axis. Then equation (6) goes over into

$$\frac{e^{-x^2/(4kt)}}{2\pi(kt)^{1/2}}\int_{-\infty}^{\infty} e^{-u_1^2}du_1.$$

This integral is the error integral and its value is $\pi^{1/2}$. Hence the product of three such integrals is

$$\frac{e^{-(x^2+y^2+z^2)/(4kt)}}{(4\pi kt)^{3/2}}.$$

Applying the faltung theorem to equation (4), we find

$$V(x, y, z, t)=\left(\frac{1}{4\pi kt}\right)^{3/2}\int_0^x\int_0^y\int_0^z f(x-x_1, y-y_1, z-z_1) \cdot e^{-(x_1^2+y_1^2+z_1^2)/(4kt)}dx_1dy_1dz_1.$$

This is the standard solution of this problem, which is usually obtained by separating variables.*

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* E. Goursat, *Cours d'Analyse*, vol. 3, p. 107, 1927.

ON A CERTAIN NON-LINEAR ONE-PARAMETER SYSTEM OF HYPERSURFACES OF ORDER n IN r-SPACE

BY B. C. WONG

Consider a linear ∞^ρ-system, where

$$\rho \leqq \frac{(n+r)!}{n!\,r!} - 1,$$

of hypersurfaces of order n, which may have σ base points, in an r-space, S_r. Denote this system by $|W|$.

Now let $\nu_1+\nu_2+\cdots+\nu_t$ projectively related curves

$$C_{11}, C_{12}, \cdots, C_{1\nu_1}, C_{21}, C_{22}, \cdots, C_{2\nu_2}, \cdots, C_{t\nu_t}$$

of orders

$$n_{11}, n_{12}, \cdots, n_{1\nu_1}, n_{21}, n_{22}, \cdots, n_{2\nu_2}, \cdots, n_{t\nu_t},$$

respectively, and all of genus p, be given in the same r-space S_r. To a point on any one of the curves corresponds a definite point on each of the other curves. We assume that none of the given curves passes through any of the σ base points of $|W|$ and that none of the intersections, if there be any, of any two of the curves is a self-corresponding point. Let $P_{11}, P_{12}, \cdots, P_{t\nu_t}$ be a set of corresponding points, the point P_{ij_i} being on the curve C_{ij_i}, $(i=1, 2, \cdots, t; j_i=1, 2, \cdots, \nu_i)$. If

$$\text{(I)} \qquad \nu_1 + 2\nu_2 + \cdots + t\nu_t = \rho \leqq \frac{(n+r)!}{n!\,r!} - 1,$$

there is one and only one hypersurface of the system $|W|$ such that $1, 2, \cdots, t$ of the points of its intersection with each of the ν_1 curves C_{1j_1}, ν_2 curves $C_{2j_2}, \cdots, \nu_t$ curves C_{tj_t}, will coincide with $P_{1j_1}, P_{2j_2}, \cdots, P_{tj_t}$, respectively. Denote such a hypersurface by V^n_{r-1}. As the corresponding points describe their respective curves, V^n_{r-1} describes a non-linear one-parameter system, $\{V\}$, of hypersurfaces of order n in S_r. In this paper we propose to determine the number, N_0, the order of the system, of the hypersurfaces of the system passing through a given

point and also the number, N_k, of those tangent to a given k-space for $k=1, 2, \cdots, r$. The symbol N_r means the number of the hypersurfaces that have each a node.

In the following, we shall give two determinations of the number N_0: the one by the use of the theory of correspondence and the other by the aid of the following known proposition.*

(A) Let there be given q varieties $V_{x_1}^{m_1}$, $V_{x_2}^{m_2}$, $\cdots$, $V_{x_q}^{m_q}$ of orders $m_1, m_2, \cdots, m_q$, respectively, such that $V_{x_i}^{m_i}$ is the locus of $\infty^1(x_i-1)$-spaces. If there exists a one-to-one correspondence between the elements of these varieties, then the locus of the $\infty^1(x_1+x_2+\cdots+x_q-1)$-spaces determined by corresponding elements is a $V_{x_1+x_2+\cdots+x_q}$ of order $m_1+m_2+\cdots+m_q$.

We now determine N_0 by the theory of correspondence. We commence with the case $\nu_1=\rho=2$, $\nu_2=\nu_3=\cdots=\nu_t=0$. The system $\{V\}$ now consists of those hypersurfaces of the net $|W|$ which pass through pairs of corresponding points on the two given curves C_{11}, C_{12}. The desired number is the number of hypersurfaces of $\{V\}$ passing through a given point, say A. Let us make a hypersurface W_{r-1}^n of $|W|$ pass through A and a point P_{11} of C_{11}. This W_{r-1}^n meets C_{12} in nn_{12} points $Q_{12}, Q_{12}', \cdots$. If one of these points happens to coincide with the point P_{12} corresponding to P_{11} on C_{11}, then W_{r-1}^n is a V_{r-1}^n of $\{V\}$. In general, this does not happen. Now pass another hypersurface $W_{r-1}'^n$ of $|W|$ through A and one of the points $Q_{12}, Q_{12}', \cdots$, say Q_{12}. This $W_{r-1}'^n$ meets C_{11} in nn_{11} points $P_{11}, P_{11}', \cdots$, to which correspond nn_{11} points $P_{12}, P_{12}', \cdots$, on C_{12}. We see that we have thus established a correspondence on the curve C_{12} such that to each of the points $Q_{12}, Q_{12}', \cdots$ correspond nn_{11} points $P_{12}, P_{12}', \cdots$ and to each of the latter correspond nn_{12} points of the former. If a united point occurs, then the two hypersurfaces W_{r-1}^n, $W_{r-1}'^n$ become coincident with a V_{r-1}^n of $|V|$. The correspondence being obviously of valence zero, the number of united points, and therefore the order of $\{V\}$, is $n(n_{11}+n_{12})$.

Suppose now $\nu_1=\rho=3$, $\nu_2=\nu_3=\cdots=\nu_t=0$. Choose a W_{r-1}^n of the web $|W|$ that passes through a given point A and a pair

* A full discussion of this proposition is found in Edge, *On the quartic developable*, Proceedings of the London Mathematical Society, (2), vol. 33, pp. 52–65. The statement above is quoted verbatim from B. C. Wong, *On the number of stationary tangent S_{t-1}'s to a V_k^n in an S_{tk+k-1}*, this Bulletin, vol. 39 (1933), pp. 608–610.

of corresponding points P_{11}, P_{12} on C_{11}, C_{12}. This W^n_{r-1} meets the third given curve C_{13} in nn_{13} points Q_{13}, Q_{13}', $\cdots$, none of which, in general, coincides with the point P_{13} corresponding to P_{11} and P_{12}. Through each of these points Q_{13}, Q_{13}', $\cdots$ there are, according to the result just found, $n(n_{11}+n_{12})$ hypersurfaces each containing A and a pair of points P_{11}, P_{12}. Then on C_{13} there are $n(n_{11}+n_{12})$ points P_{13}, P_{13}', $\cdots$ corresponding to as many pairs on C_{11}, C_{12}. Now we have on C_{13} an $(n_{11}n+n_{12}n, n_{13}n)$-correspondence of valence zero between the points P_{13}, P_{13}', $\cdots$ and the points Q_{13}, Q_{13}', $\cdots$. The number of united points, and therefore the number of the hypersurfaces of $|W|$ passing through A and a set of corresponding points on C_{11}, C_{12}, C_{13}, is $n(n_{11}+n_{12}+n_{13})$.

If we continue reasoning in this manner, we shall find that, for the case $\nu_1=\rho$, $\nu_2=\nu_3=\cdots=\nu_t=0$, the order of $\{V\}$ is $n(n_{11}+n_{12}+\cdots+n_{1\nu_1})$ or $n\sum_{j_1=1}^{\nu_1}n_{1j_1}$.

Now suppose $\nu_1=\rho-2$, $\nu_2=1$. Then the system $\{V\}$ consists of all those hypersurfaces of the ∞^ρ-system $|W|$ which contain a set of corresponding points P_{1j_1}, $(j_1=1, 2, \cdots, \nu_1)$, on the ν_1 curves C_{1j_1} and have a contact with C_{21} at the point P_{21} corresponding to P_{1j_1}. Select a W^n_{r-1} of $|W|$ passing through a given point A and a fixed set of points P_{1j_1} and having a point of contact with C_{21}. Since a pencil of hypersurfaces of order n contains $2(n_{21}n-1+p)$ members tangent to a given curve of order n_{21} and genus p, there are $2(n_{21}n-1+p)$ such hypersurfaces and hence there are as many points of contact T_{21}, T_{21}', $\cdots$ on C_{21}. None of these, in general, coincides with P_{21}. Now pass a hypersurface W'^n_{r-1} of $|W|$ through the points P_{1j_1} on the curves C_{1j_1} tangent to C_{21} at one of the points T_{21}, T_{21}', $\cdots$, say T_{21}. There are, according to the result of the preceding paragraph, $n\sum_{j_1=1}^{\nu_1}n_{1j_1}$ such hypersurfaces giving rise to as many sets of corresponding points P_{1j_1}, $P_{1'j_1}$, $\cdots$, to which correspond as many points P_{21}, P_{21}', $\cdots$, on C_{21}. Thus, we have established an $[n\sum_{j_1=1}^{\nu_1}n_{1j_1}, 2(n_{21}n-1+p)]$-correspondence also of valence zero between the points P_{21}, P_{21}', $\cdots$, and the points T_{21}, T_{21}', $\cdots$, on C_{21}. The number of united points in this correspondence which gives the order of $\{V\}$ is therefore $n\sum_{j_1=1}^{\nu_1}n_{1j_1}+2(n_{21}n-1+p)$.

These particular cases are sufficient to indicate the method used. Reasoning in exactly the same manner for all the differ-

ent values of the ν's satisfying (I), we find the general result

$$(1) \quad N_0 = \sum_{j_1=1}^{\nu_1} n_{1j_1}n + 2\sum_{j_2=1}^{\nu_2}(n_{2j_2}n - 1 + p)$$

$$+ 3\sum_{j_3=1}^{\nu_3}(n_{3j_3}n - 2 + 2p) + \cdots + t\sum_{j_t=1}^{\nu_t}[n_{tj_t}n + (t-1)(p-1)],$$

or

$$N_0 = \sum_{i=1}^{t} i\sum_{j_i=1}^{\nu_i}[n_{ij_i}n + (i-1)(p-1)],$$

where $i[n_{ij_i}n+(i-1)(p+1)]$ is the number of hypersurfaces of order n of an ∞^{i-1}-system of hypersurfaces such that i of the points of intersection of each of them with a given curve of order n_{ij_i} are coincident.

Now we determine N_0 by the aid of (A). Let the hypersurfaces of $|W|$, which may have σ base points, represent upon S_r an r-dimensional variety $\Phi_r{}^{n^r-\sigma}$ of order $n^r-\sigma$ in a ρ-space S_ρ. The $\nu_1+\nu_2+\cdots+\nu_t$ given curves, none of which is supposed to pass through any of the σ base points, are the images of curves Γ_{ij_i} of order $n_{ij_i}n$ on $\Phi_r{}^{n^r-\sigma}$ whose points are also in a one-to-one correspondence. Let R_{11}, R_{12}, $\cdots$ be a set of corresponding points, the point R_{ij_i} being on the curve Γ_{ij_i}. Corresponding to a hypersurface V_{r-1}^n of the system $\{V\}$ is a section $\Theta_{r-1}^{n^r-\sigma}$ of $\Phi_r^{n^r-\sigma}$ by a $(\rho-1)$-space which contains a set of points R_{1j_1} on the curves Γ_{1j_1}, a set of tangent lines at the points R_{2j_2} on the curves Γ_{2j_2}, a set of osculating planes at the points R_{3j_3} on the curves Γ_{3j_3}, $\cdots$. The ∞^1 $(\rho-1)$-spaces of the nature just described form an ∞^1-system to which corresponds our system $\{V\}$ of hypersurfaces. By applying (A) we find that the order of the system of $(\rho-1)$-spaces is, since the i-dimensional developable of the curve Γ_{ij_i} is of order $i[n_{ij_i}n+(i-1)(p-1)]$, the same as (1). Now through a given point A' which may be, without loss of generality, placed upon $\Phi_r{}^{n^r-\sigma}$, pass the same number of $(\rho-1)$-spaces of the system and each such $(\rho-1)$-space intersects $\Phi_r{}^{n^r-\sigma}$ in a $\Theta_{r-1}^{n^r-\sigma}$ passing through A' to which corresponds a V_{r-1}^n of $\{V\}$ passing through a given point A, the image of A'. Thus, the determination is complete.

Hitherto we have assumed that none of the given curves

passes through any of the base points of $|W|$ and that none of the intersections, if there be any, of any two of the curves is a self-corresponding point. If, however, a curve C_{ii_i} passes through one of the base points, we must deduct i, and if any two whatever of the curves intersect in a self-corresponding point, we must deduct unity from the general value of N_0 which we have just derived.

As an example consider a linear ∞^6-system $|K|$ of quartic curves in a plane ϕ with 8 base points. Let three projectively related cubic curves γ^3, γ'^3, γ''^3 of genus unity be given in the plane, none of the intersections of the curves being a self-corresponding point. Select a quartic of $|K|$ such that one of its intersections with γ^3 coincides at P, two of its intersections with γ'^3 coincide at P', and three of its intersections with γ''^3 coincide at P'', where P, P', P'' are a set of corresponding points. There are ∞^1 such quartic curves forming a non-linear pencil, $\{C\}$. Now the quartics of $|K|$ represent upon ϕ a surface Φ^8 of order 8 in S_6 upon which lie three projectively related curves Γ^{12}, Γ'^{12}, Γ''^{12} all of order 12 and genus 1, of which γ^3, γ'^3, γ''^3 are the images in ϕ. The locus of tangent lines of Γ'^{12} is of order 24 and the locus of osculating planes of Γ''^{12} is of order 36. Let R, R', R'' be a set of corresponding points on the curves. Then we say that the tangent t' to Γ'^{12} at R' and the osculating plane π'' to Γ''^{12} at R'' correspond to the point R on Γ^{12}. The 5-spaces determined by R, t', π'' will describe an ∞^1-system of 5-spaces such that $N_0=72$ of them pass through a given point A' which may be placed on Φ^8. Therefore the system $\{C\}$ of quartic curves contains 72 members passing through a given point A of ϕ.

Suppose the curve γ^3 passes through a base point of $|K|$. Then the corresponding curve Γ^{12} on Φ^8 is composed of a Γ^{11} and a line. Discarding the line or deducting unity, we have $N_0=71$. If γ'^3 alone contains a base point, the corresponding curve Γ'^{12} degenerates into a line, to be disregarded, and a Γ'^{11} whose developable surface is of order 22. Therefore, we deduct 2 and now $N_0=70$. Finally, let γ''^3 alone go through a base point. The curve Γ''^{12} on Φ^8 is made up of a line, also to be disregarded, and a Γ''^{11} the locus of whose osculating planes is of order 33. Deducting 3, we now have $N_0=69$.

Now let one of the intersections of γ^3, γ'^3 be a self-correspond-

ing point. Then Γ^{12}, Γ'^{12} also have a self-corresponding point $R \equiv R'$ to which corresponds the point R'' on Γ''^{12}. There is a linear pencil of 5-spaces passing through the tangent line t' at the self-corresponding point $R \equiv R'$ and containing the osculating plane of Γ''^{12} at R''. Disregarding this pencil, we deduct 1 and the result is $N_0 = 71$. If γ'^3, γ''^3 have a self-corresponding point in common, then one of the intersections of Γ'^{12}, Γ''^{12} is a self-corresponding point $R' \equiv R''$ to which corresponds a point R of Γ^{12}. There is a linear pencil of 5-spaces determined by R and the tangent line to Γ'^{12} at R' and the osculating plane of Γ''^{12} at $R'' \equiv R'$. Deducting 1, we have $N_0 = 71$.

We shall next proceed to determine N_k, the number of the hypersurfaces of $\{V\}$ tangent to a given k-space in S_r. We find it convenient to use the following method. We set up a one-to-one correspondence between the points of a ρ-space S_ρ and the hypersurfaces of the ∞^ρ-system $|W|$. Corresponding to the $\infty^{\rho-1}$ hypersurfaces of $|W|$ that pass through a given point A are the $\infty^{\rho-1}$ points of a $(\rho-1)$-space $S_{\rho-1}$ of S_ρ, and corresponding to the hypersurfaces of $\{V\}$ are the points of a curve Δ. Since there are given by (1), as we have seen, N_0 hypersurfaces of $|W|$ passing through A and belonging to $\{V\}$, there must be N_0 points of S_ρ common to $S_{\rho-1}$ and Δ. Hence Δ is of order N_0.

Let a k-space S_k be given in S_r. Contact being one condition, there are $\infty^{\rho-1}$ hypersurfaces of $|W|$ tangent to S_k, and to these contact hypersurfaces correspond $\infty^{\rho-1}$ points of a locus $\Sigma^M_{\rho-1}$ in S_ρ. By the methods of analytic geometry we find without difficulty that the order M of $\Sigma^M_{\rho-1}$ is $M = (k+1)(n-1)^k$. All those hypersurfaces of $|W|$ belonging to $\{V\}$ and tangent to S_k are given by all those points of S_ρ common to Δ^{N_0} and $\Sigma^M_{\rho-1}$. Therefore, the number of hypersurfaces of $\{V\}$ tangent to S_k is the number of the points in which Δ^{N_0} intersects $\Sigma^M_{\rho-1}$ and is therefore equal to $N_k = MN_0 = (k+1)(n-1)^k N_0$.

For $k = 1, 2$, then, $N_1 = 2(n-1)N_0$ and $N_2 = 3(n-1)^2 N_0$ are, respectively, the number of hypersurfaces of $\{V\}$ tangent to a given line and the number of those tangent to a given plane. If $k = r$, we have $N_r = (r+1)(n-1)^r N_0$ members of the system that have each a node.

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PRIME ENDS AND INDECOMPOSABILITY*

BY N. E. RUTT†

1. *Introduction.* Under some circumstances the set of the prime ends‡ of a plane bounded simply connected domain includes one which contains all the boundary points of the domain. This paper will establish a sufficient condition for the existence of such a prime end, and also a necessary condition. One of the arguments to be given shortly will require a special sequence of simple closed curves. The following informal development will establish satisfactorily the existence and chief properties of the sequence.

Let γ be a simply connected bounded plane domain containing the continuum G and having boundary Γ. Let ϵ be any preassigned positive number. There exists a class G_1^ϵ of simple closed curves such that if C is any one of G_1^ϵ and c is any point of C, then§ $\gamma \supset i(C) \supset G$, and $d(c, \Gamma) < \epsilon$. There exists a subclass G_2^ϵ of G_1^ϵ such that if C is any one of G_2^ϵ and g_u is any point of $\Gamma + \gamma \cdot e(C)$ and g_v any point of γ such that $d(g_v, \Gamma) > \epsilon$, then $d(g_u, C) < \epsilon$ and $i(C) \supset g_v$. If $[p_h]$, $(h = 1, 2, \cdots, m)$, is any finite subset of the points of Γ, there is a subclass G_3^ϵ of G_2^ϵ such that, if C is any one of G_3^ϵ, then $e(C) \supset \sum p_h$. There is a subclass G_4^ϵ of G_3^ϵ such that if C is any one of G_4^ϵ, then $C \cdot \Gamma$ is a finite nonvacuous set of points $[c_g]$, $(g = 1, 2, \cdots, n)$, and if K is any component of $C - \sum c_g$, then $d(K) < \epsilon$. If $[P_{hk}]$, $(h = 1, 2, \cdots, m$ and $k = 1, 2)$, is a set of $2m$ arcs with end points $[p_{hk}, c_{hk}]$, respectively, such that $\gamma \supset \sum (P_{hk} - p_{hk})$, and if $p_{hk} = p_h$, $(k = 1, 2)$, for all values of h, and if $\gamma \cdot P_{hk} \cdot P_{h'k'} = 0$ for all admissible values of h, k, h', and k' except of course when $h = h'$ and $k = k'$;

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† Theorems 1 and 2 of this paper were obtained while the author was holding a National Research Fellowship.

‡ Defined by C. Carathéodory in his paper, *Über die Begrenzung einfach zusammenhängender Gebiete*, Mathematische Annalen, vol. 73 (1912), pp. 323–370.

§ If K is a point set then $c(K)$ represents its complement in space. If C is a simple closed curve in the plane then $i(C)$ represents its interior and $e(C)$ its exterior. Combining the symbols gives $ce(C) = C + i(C)$ and $ci(C) = C + e(C)$.

then there exists a subclass G_5^ϵ of G_4^ϵ such that, if C is any one of G_5^ϵ, then $i(C) \supset \sum c_{hk}$ and $C \cdot \sum P_{hk}$ is exactly $2m$ points. There is a subclass G_6^ϵ of G_5^ϵ such that, if C is any element of G_6^ϵ and K is any component of $C - \sum c_g$ containing a point of $\sum P_{hk}$, then K contains exactly two such points and is not adjacent on C to any component of $C - \sum c_g$ containing any points of $\sum P_{hk}$. It is to be observed that when $[p_{hk}]$ is vacuous, then $G_2 = G_3$ and $G_4 = G_5 = G_6$.

The existence of G_6^ϵ implies the existence of systems of simple closed curves now to be defined and hereafter to be referred to as of *type W* with respect to γ and G. When γ is any plane bounded connected and simply connected domain containing the continuum G and having boundary Γ, there exists an infinite sequence $[C_i]$, $(i = 1, 2, 3, \cdots)$, of simple closed curves related to a corresponding infinite sequence of positive numbers λ_i, $(\lambda_{i+1} < \lambda_i/2)$, in such a way that for each value of j: (1) the domain $i(C_j) \supset G$; (2) the domain $\gamma \supset i(C_j)$; (3) the set $\Gamma \cdot C_j$ is a nonvacuous finite point set $[c_{ji}]$, $(i = 1, 2, \cdots, n_j)$, and $\sum_{i=1}^{n_j} c_{ji} \cdot \sum_{m=1}^{j-1} \sum_{i=1}^{n_m} c_{mi} = 0$; (4) the components of $C_j - \sum_{i=1}^{n_j} c_{ji}$ are a set $[A_{ji}]$ of cuts of γ each of diameter less than λ_j; (5) the components of $C_m \cdot ci(C_j)$, $(m = 1, 2, \cdots, j-1)$, are a set of n_m mutually exclusive arcs $[C_{mi}^j]$ each of which contains precisely one point of $[c_{mi}]$, and $\sum_{i=1}^{n_h} C_{hi}^j \cdot \sum_{i=1}^{n_k} C_{ki}^j = 0$ if h and k are unequal values of m; (6) no two successive elements of the set of cuts $[A_{ji}]$ contain points of $\sum_{m=1}^{j-1} \sum_{i=1}^{n_m} C_{mi}^j$; (7) for any i, if $A_i \cdot \sum_{m=1}^{j-1} \sum_{i=1}^{n_m} C_{mi}^j \neq 0$, it is two points; and (8) if R is a point set such that $\gamma \cdot R \neq 0$, a number r exists such that, if $j > r$, then $i(C_j) \cdot R \neq 0$.

The existence of the sequence may be seen as follows. With $[p_h]$ vacuous, at least two points in $[c_g]$, and $\lambda = \lambda_1$ an arbitrary positive number, let C_1 be any element of G_6^ϵ. Assume that all curves of the set $[C_i]$ up to and including C_s have been constructed so as to possess as a set properties (1) to (7) inclusive. The next curve of the set will be constructed. Let $[p_h]$ be the set $\sum_{j=1}^{s} \sum_{i=1}^{n_j} c_{ji}$, $(h = 1, 2, \cdots, m$ and $m = \sum_{i=1}^{s} n_i)$. Let σ be the least positive number of the form $d(p_u, p_v)$, (u and $v \leqq h$). Let S_h, $(h = 1, 2, \cdots, m)$, be a circle with center p_h and radius less than $\sigma/3$ intersecting exactly one of the curves $C_1, C_2, \cdots, C_s$. For each value of h let P_h be the arc of $ce(S_h) \cdot \sum_{i=1}^{m} C_i$ which is maximal and contains p_h. Let η be $d(\Gamma, \sum_{i=1}^{s} C_i - \sum_{i=1}^{m} P_h)$.

Finally, with λ_{s+1} chosen less than $\lambda_s/2$ and less than η, and $[P_{hk}]$, $(h=1, 2, \cdots, m$ and $k=1, 2)$, chosen so that, for each value of h, the arc $P_h=P_{h1}+P_{h2}$, where P_{h1} and P_{h2} are arcs with only p_h in common, let C_{s+1} be any element of G_δ^τ, $(\tau=\lambda_{s+1})$. The curves $C_1, C_2, \cdots, C_s, C_{s+1}$ have now been constructed in such a way as to possess all except the last of the properties listed. By continuation of the process an infinite sequence results which clearly has property (8).

2. *Sufficient Condition.* LEMMA 1. *Suppose* (1) *that* γ *is a plane bounded connected and simply connected domain containing a continuum* G *and having boundary* Γ, (2) *that* $[C_i]$ *is a collection of simple closed curves of type* W *with respect to* γ *and* G, *the general element* C_j *of which consists of the finite nonvacuous subset* $[c_{ji}]$ *of the points of* Γ *and the set* $[A_{ji}]$ *of cuts of* γ, (3) *that* $[A_i]$ *is any chain of cuts of* γ *each one of which for some value of* j *is included in the collection* $[A_{ji}]$, *and* (4) *that* $\mathfrak{P}$ *is a prime end of* γ *and is defined by the chain of cuts* $[B_i]$ *of* γ, *where there is a point of* Γ *every neighborhood of which contains all but a finite number of* $[B_i]$. *Then* (1) *the chain* $[A_i]$ *defines a prime end of* γ, *and* (2) *there is a chain of cuts of* γ *defining* $\mathfrak{P}$ *each one of which for some value of* j *belongs to* $[A_{ji}]$.

As $[A_i]$ is a chain, no more than one element of it can belong to the collection $[A_{ji}]$ for any given value of j. Thus $\lim_{i\to\infty} d(A_i)=0$, and so $[A_i]$ defines a prime end of γ.

On the other hand, owing to the conditions imposed upon $[B_i]$, there is a point b such that any domain containing b contains all but a finite number of $[B_i]$. There is a positive integer n_0 such that if $j>n_0$, then $C_j \not\ni b$. Accordingly there is a neighborhood η_0 of b containing no point of $ce(C_{n_0})$ and within it an element B_{n_0} of $[B_i]$. The cut B_{n_0} is separated in γ from $i(C_{n_0})$ by an element $A_{n_0 b}$ of $[A_{n_0 i}]$. There is a positive number ρ_0 such that $\rho_0<d(B_{n_0}, B_{n_0+1})$ and a subscript n_1, $\lambda_{n_1}<\rho_0$, such that $i(C_{n_1})$ contains points of B_{n_0+1}. As no element of $[A_{n_1 i}]$ has points in both B_{n_0} and B_{n_0+1}, there are elements of $[A_{n_1 i}]$ separated from $i(C_{n_0})$ in γ by B_{n_0}. There is a neighborhood η_1 of b such that $ce(C_{n_1})\cdot\eta_1=0$ and $\eta_1 \supset B_{n_1}$, where $n_1>n_0+1$. But then $e(C_{n_1}) \supset B_{n_1}$, so there is an element $A_{n_1 b}$ of $[A_{n_1 i}]$ which separates B_{n_1} and $i(C_{n_1})$ in γ, and this one separates B_{n_1} and B_{n_0} in γ, for it has no point in B_{n_0+1}. The process may be con-

tinued indefinitely, and the resulting chains of cuts B_{n_0}, $B_{n_1}, \cdots$ and $A_{n_0 b}, A_{n_1 b}, \cdots$ are equivalent. Thus there is a chain of cuts of γ defining $\mathfrak{P}$, any one of whose elements is included for some value of j in the collection $[A_{ji}]$.

THEOREM 1. *In order that the collection of the prime ends of the plane bounded connected and simply connected domain γ having boundary Γ should include one containing Γ, it is sufficient that Γ be indecomposable.*

Let G be any point of γ and $[C_i]$ be a collection of simple closed curves of type W with respect to γ and G. For every value of j the set $\gamma - \gamma \cdot \sum_{i=1}^{j} ce(C_i)$ is a finite set of domains $[\gamma_{ji}]$, some separated from G by one of $[A_{ji}]$ and the others by a cut which is the sum of a subarc of an element of $[A_{ji}]$ and a subarc of some other one of $[C_i]$. It is to be noted that each one of $[\gamma_{j+1,\,i}]$ is contained in one of $[\gamma_{ji}]$, that the boundary of the element γ_{ji} has as its subset in Γ a continuum Γ_{ji}, that $\Gamma = \sum \Gamma_{ji}$ for each value of j, and that if the subcollection of $[\gamma_{j+1,i}]$ contained in γ_{jk} is $[\gamma^{k}_{j+1,i}]$, then $\Gamma_{jk} = \sum \Gamma^{k}_{j+1,i}$.

Now as Γ is indecomposable, there is a specific element Γ^{q}_{1} of $[\Gamma_{1i}]$ which is identical with Γ. Moreover as $\Gamma^{q}_{1} = \sum \Gamma^{q}_{2i}$ there is a specific element Γ^{q}_{2} of $[\Gamma^{q}_{2i}]$ which is identical with Γ, and so on indefinitely. There arises in this way an infinite sequence of domains $\gamma_1, \gamma_2, \gamma_3, \cdots$, where for each value of j the domain γ_j is an element of $[\gamma_{ji}]$, the domain $\gamma_j \supset \gamma_{j+1}$, and $\gamma'_j = \Gamma_j + H_j$, where H_j is a cut of γ, if not contained in a single one of $[C_i]$, then the sum of an arc from one of $[C_i]$ and an arc from another.

If for two different subscripts, u and v, $(u<v)$, the sets $H_u \cdot C_j$ and $H_v \cdot C_j$ are nonvacuous (j being fixed), then, when $u \leqq m \leqq v$, the set $H_m \cdot C_j \neq 0$, since indeed $H_u \cdot C_j \supset H_m \cdot C_j \supset H_v \cdot C_j$. Accordingly if ($j$ being fixed), for indefinitely many values of u, the set $H_u \cdot C_j \neq 0$, then $H_u \cdot C_j \neq 0$ for all values of u subsequent to j. But in this case when $u > j$ the set $H_u \cdot e(C_j)$ belongs to an element B_u of $[A_{ui}]$. Moreover when $u > j$ the collection $[B_u]$ is a set of mutually exclusive arcs defining a prime end $\mathfrak{P}$ of γ. Since, for each of these values of u, the cut B_u separates γ_u from G, the prime end $\mathfrak{P}$ contains Γ.

On the other hand, if regardless of the value of j the set $H_u \cdot C_j \neq 0$ for at most a finite set of values of u, then beginning with any element H_{j_1} of $[H_j]$, there exists an element H_{j_2}

subsequent to H_{j_1} having no points in the members of $[C_i]$ in which H_{j_1} has points, an element H_{j_2} subsequent to H_{j_2} having no points in the members of $[C_i]$ in which $H_{j_1}+H_{j_2}$ has points, and so on indefinitely. Let B_n be an element of $[A_{j_n i}]$ in which H_{j_n} has points. Then $[B_n]$ is a collection of mutually exclusive cuts of γ forming a chain and defining a prime end $\mathfrak{P}$ of γ. Moreover as B_n separates G from γ_{j_n}, the prime end $\mathfrak{P}$ contains Γ.

3. *Necessary Condition.* LEMMA 2. *Suppose* (1) *that γ is a plane bounded connected and simply connected domain with boundary Γ,* (2) *that $\mathfrak{P}$ is a prime end of γ containing the subset P of Γ and defined by a chain of domains $[\eta_i]$ determined respectively by a chain of cuts $[H_i]$ of γ where there is only one point of Γ every neighborhood of which contains points of more than one of $[H_i]$, and* (3) *that Q is any proper subcontinuum of P. Then the set of the points of $(\Gamma-Q)\cdot\sum\overline{H}_i$ is infinite.*

Only the case in which $(\Gamma-Q)\cdot\sum\overline{H}_i$ is vacuous needs to be considered. Let δ be the complementary domain of $c(Q)$ containing all but the ends of each of $[H_i]$. Each of $[H_i]$ is then a cut of δ. Of the two domains into which H_1 divides δ, let δ_1 be the one containing all but the ends of H_2. Clearly then $\delta_1 \supset \eta_1$. But H_2 is a cut of δ_1; of the two domains into which H_2 cuts δ_1, let δ_2 be the one which contains all except the ends of H_3. Then $\delta_2 \supset \eta_2$. In general, for $n=3, 4, 5, \cdots$, clearly H_n is a cut of δ_{n-1}. Of the two domains into which H_n divides δ_{n-1}, let δ_n be the one which contains H_{n+1}. Thus $\delta_n \supset \eta_n$ for each value of n. It is readily seen that at most a finite number of $[\delta_i]$ are unbounded. Consider the sets $\prod\bar{\delta}_i$ and $\prod\bar{\eta}_i$. Since, for each value of i, $\delta_i \supset \eta_i$, then $\prod\bar{\delta}_i \supset \prod\bar{\eta}_i$. Now by supposition $\prod\bar{\eta}_i$ is the point set contained by $\mathfrak{P}$, that is, it is P, whereas $\prod\bar{\delta}_i \subset Q$. Thus the assumption that $(\Gamma-Q)\cdot\sum H_i$ is vacuous implies the contradiction $Q \supset P$.

LEMMA 3. *Suppose* (1) *that γ is a plane bounded connected and simply connected domain with boundary Γ, and* (2) *that $\mathfrak{P}$ is a prime end of γ containing the continuum P of Γ whose continuum of chief points is P_c, and* (3) *that Q is a proper subcontinuum of P contained in P_c. Then Q is a continuum of condensation of Γ.*

Let q be any point of Q. As q is a chief point of $\mathfrak{P}$ there is a

chain of cuts $[H_i]$ of γ defining $\mathfrak{P}$, such that every neighborhood of q contains points of all but a finite number of $[H_i]$ and $\lim_{i\to\infty} d(H_i)=0$. Thus q is a limit point of every infinite subset of the end points of the elements of $[H_i]$. Since Q is a proper subset of P, the point set $(\Gamma-Q)\cdot\sum\overline{H}_i$ is infinite, and q is a limit point of this set. Thus $\overline{\Gamma-Q}\supset q$, as was to be proved.

THEOREM 2. *In order that the plane bounded connected and simply connected domain γ having boundary Γ should have a prime end $\mathfrak{P}$ containing Γ it is necessary that Γ should be either indecomposable or the sum of two indecomposable continua.*

Suppose that the prime end $\mathfrak{P}$ of γ contains Γ. Let $\mathfrak{P}$ be defined by the chain of cuts $[H_i]$ of γ where there is just one point of Γ every neighborhood of which contains points of more than one of $[H_i]$ and $\lim_{i\to\infty} d(H_i)=0$. Let P_c be the continuum of the chief points of $\mathfrak{P}$. Suppose first that $P_c=\Gamma$, and let Q be any proper subcontinuum of Γ. Then Q is a continuum of condensation of Γ by Lemma 3, and so Γ is indecomposable.

Suppose next that $P_c\neq\Gamma$, and that Γ is decomposable and the sum of D_0 and D_3, two proper subcontinua of it. The set P_c is thus a continuum of condensation of Γ. If $P_c\supset D_0$ then $D_3\supset\Gamma-P_c$, and $D_3\supset\overline{\Gamma-P_c}=\Gamma$, which would mean that $D_3\equiv\Gamma$; thus $P_c\not\supset D_0$ and $P_c\not\supset D_3$. In addition $P_c\cdot D_0\neq 0$ and $P_c\cdot D_3\neq 0$, for if it is supposed that one of these is vacuous then Lemma 2 is contradicted, as all but a finite number of $[H_i]$ will necessarily have both of their end points in D_0 or D_3 in consequence. Accordingly $\Gamma-P_c\not\subset D_0$, and so $\Gamma-(P_c+D_0)=Q_3\neq 0$, where $D_3\supset Q_3$, and similarly $\Gamma-(P_c+D_3)=Q_0\neq 0$, where $D_0\supset Q_0$.

Suppose now that D_0 is expressible as the sum of its two proper subcontinua D_1 and D_2. Consider the possibility that neither D_1 nor D_2 contains Q_0. In this case $\Gamma-(P_c+D_1+D_2)$ is the nonvacuous set Q_3, also $\Gamma-(P_c+D_1+D_3)$ is a nonvacuous subset Q_2 of D_2, and $\Gamma-(P_c+D_2+D_3)$ is a nonvacuous subset Q_1 of D_1. As $D_0\cdot P_c\neq 0$, either D_1 or D_2 contains a point of P_c. If $D_1\cdot P_c=0$, then D_2+D_3 is a continuum containing P_c and a contradiction of Lemma 2 speedily results. Thus $D_1\cdot P_c\neq 0$ and $D_2\cdot P_c\neq 0$. But then, either when one of the three continua $D_1+P_c+D_2$, $D_1+P_c+D_3$, and $D_2+P_c+D_3$, or when none of the three, contains both ends of infinitely many of $[H_i]$, there

follows a contradiction of Lemma 2, so it is not possible for neither D_1 nor D_2 to contain Q_0.

Suppose D_0 replaced by D_1, where D_1 is a subcontinuum of D_0 irreducible about Q_0. Then $\Gamma = D_1 + D_3$, and the properties of D_0 and D_3 already derived are possessed also by D_1 and D_3. But if D_1 is decomposable, then neither of any two of its proper subcontinua of which it is the sum can contain Q_0, as D_1 is irreducible about Q_0, whereas, if neither of them contains Q_0, then the contradiction above appears.

So D_1 is indecomposable, and if D_2 is a subcontinuum of D_3 irreducible about $\Gamma - (P_c + D_1)$, then D_2 is also indecomposable. As $\Gamma = D_1 + P_c + D_2$, then $\Gamma - P_c = (D_1 + D_2) - (D_1 + D_2) \cdot P_c$; and as $\overline{\Gamma - P_c} = \Gamma$, then $\overline{(D_1 + D_2) - (D_1 + D_2) \cdot P_c} = \Gamma$. This means that $D_1 + D_2 \supset P_c$, so $\Gamma = D_1 + D_2$, and the theorem is proved.

4. *Remarks and Generalizations.* The sufficient condition of §2 is not necessary, as the following example shows. Let the curvilinear quadrilateral K_1 with vertices P, Q, R, and S consist of the points (x, y) where $x \geqq 0$ and $1 \leqq x^2 + y^2 \leqq 4$; and the curvilinear quadrilateral K_2, also with vertices P, Q, R, and S, consist of the points (x, y) where $x \leqq 0$ and $1 \leqq x^2 + y^2 \leqq 4$. In each case let the four vertices be $P(0, 2)$, $Q(0, 1)$, $R(0, -1)$, and $S(0, -2)$. Upon the square $A(0, 0)$, $B(0, 1)$, $C(1, 1)$, and $D(1,0)$ suppose the indecomposable continuum of Brouwer* inscribed. Let K_2 be homeomorphic with $ABCD$ in such a way that D coincides with Q, C with P, B with S, and A with R. Also let K_1 be symmetric to K_2 with respect to the y axis. A domain γ results containing the origin of coordinates and bounded by a bounded decomposable continuum Γ which is the sum of two of Brouwer's indecomposable continua. The set contains PQ, and PQ belongs to a prime end of γ which contains Γ.

The necessary condition of §3 is not sufficient, as the following example shows. Let K_2 be homeomorphic with $ABCD$ as above. Let K_1 be homeomorphic with $ABCD$ also, in such a way that A coincides with Q, B with P, C with S, and D with R. A bounded domain results having a boundary which is the

* As in the figure on p. 228 of vol. 8 of the Fundamenta Mathematicae. See also Brouwer, Mathematische Annalen, vol. 68, p. 423. Brouwer's diagram includes more than is used in the examples of this paper.

sum of two of Brouwer's indecomposable continua, and no prime end of this domain contains both of these continua.

It may be remarked that Lemma 2 is not true if $\lim_{i\to\infty} d(H_i)\neq 0$, and neither Lemma 2 nor Lemma 3 is true if Q is not a proper subcontinuum of P. Lemma 3 implies the following theorem, of interest in itself, which was proved in the derivation of Theorem 2.

THEOREM 3. *If the plane bounded connected and simply connected domain γ has a prime end whose set of chief points coincides with Γ, the boundary of γ, then Γ is indecomposable.*

The theorem of §2 may be generalized.

LEMMA 4. *Suppose* (1) *that the point set D is a plane bounded indecomposable continuum,* (2) *that the point set K is a plane bounded continuum such that $D\cdot c(K)\neq 0$, and that $\overline{D\cdot c(K)}\neq D$, and* (3) *that γ is a component of $c(D+K)$. Then $\overline{\gamma}\not\supset D$.*

Suppose indeed that γ is a component of $c(D+K)$ such that $\overline{\gamma}\supset D$. Let δ be the component of $c(K)$ which contains γ. Then $\delta\cdot D=c(K)\cdot D$ and is a non-vacuous set. Since $\overline{D\cdot c(K)}\neq D$, the set $\delta\cdot D$ can not contain entirely any of the composants of D. Consequently $\delta\cdot D$ is a collection of components no one of which contains points of more than one composant of D. Thus the components of $D\cdot c(K)$ are an uncountable collection $[D_\alpha]$, and as $\overline{\gamma}\supset\sum D_\alpha$, this collection is orderable.*

It is moreover readily seen that no two of $[D_\alpha]$ belonging to one composant of D can separate two which belong to some other composant, so that the subcollection of $[D_\alpha]$ included in any composant of D is an interval of the collection $[D_\alpha]$. Let D_k be any element of $[D_\alpha]$ and contain the point k. The point k is a limit of every composant of D, and so is a limit of each of an uncountable set of series of $[D_\alpha]$ each contained in a different composant of D. This is a contradiction, as the set of series in question is uncountable in number and as no two of them are concurrent.†

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* N. E. Rutt, *On certain types of plane continua*, Transactions of this Society, vol. 33 (1931), p. 806.

† N. E. Rutt, *Concurrence and uncountability*, this Bulletin, vol. 39 (1933), Corollary 1, p. 299.

THEOREM 4. *If the boundary Γ of the plane bounded connected and simply connected domain γ contains an indecomposable continuum D, there is a prime end of γ which contains D.*

Here, as in the development of Theorem 2, for each value of j the set $\Gamma=\sum\Gamma_{ji}$. Consequently $\sum\Gamma_{ji}\supset D$. If for each of these $\overline{c(\Gamma_{ji})\cdot D}\supset D$, then the set $\sum\Gamma_{ji}\cdot D$ is nowhere dense in D and $[\Gamma_{ji}]$ does not cover D. But as none of $[\Gamma_{ji}]$ can have $\overline{c(\Gamma_{ji})\cdot D}\neq D$ unless $D\cdot c(\Gamma_{ji})=0$, in view of Lemma 4, there must for every value of j be one of $[\Gamma_{ji}]$ which contains D. The proof now follows lines almost identical with those of Theorem 2.

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PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVES

BY L. R. WILCOX

In a fundamental paper* on the projective differential geometry of curves, L. Berzolari obtained canonical expansions representing a curve C immersed in a linear space S_n in a neighborhood of one of its points P_0. The vertices of the coordinate simplex yielding Berzolari's canonical form are covariantly related to the curve, while the unit point may be any point of the rational normal curve Γ which osculates C at P_0. It is the purpose of the present paper to define a covariant point on Γ which can be chosen as a unit point so as to produce final canonicalization of the power series expansions of Berzolari.

It will be observed that the usual methods of defining a point on Γ for the cases $n=2$ and $n=3$ depend on configurations† that do not possess suitable analogs in n-space. Hence it appeared for some time that the problem called for different procedures in spaces of different dimensionality. Special devices

* L. Berzolari, *Sugli invarianti differenziali proiettivi delle curve di un iperspazio*, Annali di Matematica, (2), vol. 26 (1897), pp. 1–58.

† E. P. Lane, *Projective Differential Geometry of Curves and Surfaces*, pp. 12–27.

were found by S. B. Murray and the author* for the spaces S_4 and S_5; however, like the methods used in the plane and in S_3 these seem not to admit generalization. It is to be shown here that, with the help of a suitably chosen linear complex, the general problem for $n>3$ may be solved.

Local power series expansions representing an analytic curve C immersed in a linear space S_n of n dimensions $(n>3)$ in a neighborhood of an ordinary point P_0 may be written† in the form,

$$(1)\qquad \begin{aligned} x_0 &= 1, \\ x_i &= x_1^i + a_i x_1^{n+2} + b_i x_1^{n+4} + \cdots, \qquad (i = 2, \cdots, n), \end{aligned}$$

wherein $x_0, \cdots, x_n$ are homogeneous projective point coordinates, and the coefficients a_i, b_i, etc. are complex numbers, a_{n-1} being zero and a_n different from zero. The equations of the osculating rational normal curve Γ of C at P_0 are

$$x_i = x_1^i, \qquad (i = 0, \cdots, n).$$

The vertices of the coordinate simplex will be denoted by $P_0, \cdots, P_n$, where P_i is the point for which

$$x_i = 1, \qquad x_j = 0, \qquad (j = 0, \cdots, n; j \neq i).$$

The point P_n is the intersection that is distinct from P_0 of the curve Γ and the principal hyperplane‡ of C and Γ; the vertex P_i, $(i=1, \cdots, n-1)$, is the intersection of the osculating space S_{n-i} of Γ at P_n and the osculating space S_i of C at P_0. The unit point $U(1, \cdots, 1)$ is any point on Γ distinct from the points P_0 and P_n.

Homogeneous line coordinates p_{ij} of the line joining points $X(x_0, \cdots, x_n)$ and $Y(y_0, \cdots, y_n)$ will be defined by

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* See Murray, *Curves in Four-Dimensional Space*, Chicago master's dissertation, 1934, and Wilcox, *Curves in Five-Dimensional Space*, Chicago master's dissertation, 1933.

† Berzolari, loc. cit., p. 2. We shall say that P_0 is an ordinary point of C in case (1) C is not hyperosculated at P_0 by any of its linear osculants or by its osculating rational normal curve Γ, and (2) C and Γ have at P_0 a principal plane not contained in their osculating hyperplane at P_0. For the definition of principal plane see Berzolari, loc. cit., p. 18.

‡ Berzolari, loc. cit., p. 19.

$$p_{ij} = x_i y_j - x_j y_i, \qquad (i, j = 0, \cdots, n;\ i < j).$$

The coordinates of the line l_{hk} joining P_h and P_k $(h<k)$ are given by

$$p_{ij} = \begin{cases} 1, & \text{when } i = h \text{ and } j = k, \\ 0, & \text{when } i \neq h \text{ or } j \neq k. \end{cases}$$

In the totality of linear complexes in the ambient space S_n there is a two-parameter family containing all lines l_{hk} except $l_{0,n}$, $l_{1,n-1}$, and $l_{2,n}$. The equation of this family is

$$\lambda p_{0,n} + \mu p_{1,n-1} + \nu p_{2,n} = 0, \tag{2}$$

wherein λ, μ, ν are homogeneous parameters. In the family (2) there is a unique complex having $(n+3)$-line contact with the tangent developable of the curve C at the line $l_{0,1}$. With the help of expansions (1) its equation is found to be

$$\begin{aligned}(n-2)(n-3)p_{0,n} - n(n-3)p_{1,n-1} \\ - (n-2)(n+3)a_n p_{2,n} = 0.\end{aligned} \tag{3}$$

The locus of all lines of the complex (3) through the point P_n is a hyperplane π whose equation is

$$(n-3)x_0 - (n+3)a_n x_2 = 0.$$

If we demand that the unit point U shall lie in this hyperplane, we have

$$a_n = \frac{n-3}{n+3};$$

hence we obtain the following result.

An analytic curve C immersed in a linear space of n dimensions may be represented in a neighborhood of one of its ordinary points P_0 by local power series expansions of the form (1), *in which* $a_n=(n-3)/(n+3)$. *For this canonical form the unit point is one of the intersections distinct from P_n of the hyperplane π with the osculating rational normal curve of C at P_0.*

UNIVERSITY OF CHICAGO

THE APPROXIMATE SOLUTION OF INTEGRAL EQUATIONS*

BY E. N. OBERG

1. *Introduction.* Consider the Fredholm integral equation

$$L(u) \equiv u(x) - \int_a^b k(x, t)u(t)dt = f(x). \tag{1}$$

Let it be assumed that the given function $f(x)$ is continuous in the interval $a \leqq x \leqq b$, and that the kernel $k(x, t)$ is continuous in the square $a \leqq x \leqq b$, $a \leqq t \leqq b$. Let us assume also that the equation $L(u) = 0$ has no non-trivial solution. Then a unique continuous solution exists for the unknown $u(x)$ of (1) of the form†

$$u(x) = f(x) + \int_a^b H(x, t)f(t)dt, \tag{2}$$

in which the resolvent kernel $H(x, t)$ is a well-determined continuous function in the square $a \leqq x \leqq b$, $a \leqq t \leqq b$.

Let

$$P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

be an arbitrary polynomial of degree n. Then a problem in minima is to determine the coefficients of this polynomial so that the integral

$$\int_a^b |f(x) - L(P_n)|^m dx \tag{3}$$

shall be a minimum, where m is any positive real number.

The purpose of this paper is to examine the existence and uniqueness of such a polynomial and its convergence towards

* Presented to the Society, June 23, 1933.

† See, for example, Courant-Hilbert, *Methoden der mathematischen Physik*, 2d ed., 1931, vol. 1, pp. 121–124. Less restrictive hypotheses can be placed on $k(x, t)$ (see, for example, E. W. Hobson, *On linear integral equations*, Proceedings of the London Mathematical Society, vol. 13 (1913–1914), pp. 307–340) but for the present discussion the hypothesis mentioned above may be regarded as sufficiently illustrative.

$u(x)$ as n becomes infinite. Various investigations of a similar nature have been conducted by Krawtchouk,* Enskog,† and Picone,‡ but by methods different from ours and in every case for $m>1$. Picone has considered the same problem as ours but only for the special case $m=2$, and by a method which does not appear capable of extension to other values of m. Further mention should also be made of a paper by McEwen§ on linear differential equations to which the following paper is in many aspects an analog.

2. *The Existence and Uniqueness of an Approximating Polynomial.* The questions of existence and uniqueness of a polynomial minimizing (3) can be disposed of readily by application of theorems which are already well known on least mth power approximation.¶ The problem can be looked upon as that of approximating the given function $f(x)$ by a linear combination of the $n+1$ continuous functions $L(1)$, $L(x)$, $L(x^2), \cdots, L(x^n)$. These functions are linearly independent in the interval $a \leqq x \leqq b$, since to assume otherwise would lead to the conclusion that a polynomial $p_n(x)$ exists, not identically equal to zero, which satisfies the homogeneous equation $L(p_n)=0$ in contradiction with the hypothesis placed on $L(u)$. It follows from the general theorems to which reference has been made that for $m>0$ a minimizing polynomial exists, and that for $m>1$ this polynomial is unique.

3. *Convergence of the Approximating Polynomial for* $m \geqq 1$. Let $P_n(x)$ be the polynomial which minimizes (3), and $u(x)$ the unique continuous solution of (1). Let $q_n(x)$ be any nth degree polynomial, and ϵ_n a corresponding upper bound for the absolute value of $r_n(x)=u(x)-q_n(x)$. Let γ_n be the minimum of (3):

* See M. Krawtchouk, *Sur la résolution approchée des équations intégrales linéaires*, Comptes Rendus (Paris), vol. 188 (1929), pp. 978–980.

† See D. Enskog, *Eine allgemeine Methode zur Auflösung von linearen Integralgleichungen*, Mathematische Zeitschrift, vol. 24 (1926), pp. 670–682.

‡ M. Picone, *Sul metodo delle minime potenze ponderate e sul metodo di Ritz*, Rendiconti del Circolo Matematico di Palermo, vol. 52 (1928), pp. 225–254.

§ See W. H. McEwen, *Problems of closest approximation connected with the solution of linear differential equations*, Transactions of this Society, vol. 33 (1931), pp. 979–997, and also this Bulletin, vol. 38 (1933), pp. 887–894.

¶ For a proof, see, for example, D. Jackson, *On functions of closest approximation*, Transactions of this Society, vol. 22 (1921), pp. 117–128.

$$\gamma_n = \int_a^b |f(x) - L(P_n)|^m dx = \int_a^b |L(u) - L(P_n)|^m dx$$

$$= \int_a^b |L(u - P_n)|^m dx. \tag{4}$$

Since $q_n(x)$ is also a polynomial of the nth degree, it follows from the minimizing property of γ_n that

$$\gamma_n = \int_a^b |L(u - P_n)|^m dx$$

$$\leqq \int_a^b |L(u - q_n)|^m dx = \int_a^b |L(r_n)|^m dx. \tag{5}$$

But

$$L(r_n) = r_n(x) - \int_a^b k(x, t) r_n(t) dt,$$

whence if $|k(x, t)| \leqq M$,

$$|L(r_n)| \leqq |r_n(x)| + \int_a^b |k(x, t)|\,|r_n(t)|\,dt$$

$$\leqq |1 + M(b - a)|\,\epsilon_n. \tag{6}$$

Hence if (6) is substituted in (5), it follows that

$$\gamma_n \leqq (N\epsilon_n)^m, \tag{7}$$

where N is a positive constant not depending on n or ϵ_n.

If the difference $P_n(x) - q_n(x)$ is denoted by $\pi_n(x)$, then $r_n - \pi_n$ is the same as $u - P_n$. Let $\phi_n(x) = r_n(x) - \pi_n(x)$, and let $z(x)$ represent the continuous function $L(\phi_n)$:

$$z(x) = \phi_n(x) - \int_a^b k(x, t)\phi_n(t) dt. \tag{8}$$

From (2),

$$\phi_n(x) = z(x) + \int_a^b H(x, t) z(t) dt,$$

and if $|H(x, t)| \leqq G$ throughout the square $a \leqq x \leqq b$, $a \leqq t \leqq b$, then

$$(9) \qquad |\phi_n(x) - z(x)| \leqq G\int_a^b |z(t)|\,dt.$$

The integrand on the right is non-negative and continuous in the closed interval (a, b), so that a form of Hölder's inequality* can be applied to (9) with the result that

$$|\phi_n(x) - z(x)| \leqq G(b-a)^{(m-1)/m}\left[\int_a^b |z(t)|^m dt\right]^{1/m}.$$

The value of the integral in the brackets is γ_n, and by virtue of (7),

$$|\phi_n(x) - z(x)| \leqq G(b-a)^{(m-1)/m}\gamma_n^{1/m} \leqq \mu\epsilon_n.$$

For convenience in the use of the notation later, it will be understood that μ represents the greater of the two quantities 1 and $G(b-a)^{(m-1)/m}N$. But from (8),

$$\left|\int_a^b k(x,t)\phi_n(t)dt\right| = |\phi_n(x) - z(x)|,$$

whence

$$(10) \qquad \left|\int_a^b k(x,t)\phi_n(t)dt\right| \leqq \mu\epsilon_n.$$

Let the maximum of $|\pi_n(x)|$ for $a \leqq x \leqq b$ be $\mu\sigma_n$, and x_0 a point in the interval where $|\pi_n(x_0)| = \mu\sigma_n$. Then from Markoff's theorem,†

$$|\pi_n'(x)| \leqq 2\mu n^2\sigma_n/(b-a)$$

for all values of x in $a \leqq x \leqq b$. For $|x - x_0| \leqq (b-a)/4n^2$, by the mean value theorem,

---

* We are applying the inequality

$$\int_a^b F(x)dx \leqq (b-a)^{(P-1)/P}\left[\int_a^b [F(x)]^P dx\right]^{1/P},$$

where $F(x)$ is assumed $\geqq 0$, and $P \geqq 1$. It is because of the fact that Hölder's inequality applies for $P \geqq 1$ only, that we must break up the convergence proof into two parts, one for $m \geqq 1$, and the other for $m < 1$, which must be approached by a different method.

† See, for example, D. Jackson, Transactions of this Society, vol. 22 (1921), p. 163.

$$|\pi_n(x) - \pi_n(x_0)| \leqq \mu\sigma_n/2,$$

from which it follows that

$$|\pi_n(x)| \geqq \mu\sigma_n/2.$$

Let it be assumed for the time being that ϵ_n is less than or at most equal to $\sigma_n/4$ (the contrary case which leads directly to the desired conclusion will be discussed later). Then

$$|\phi_n(x)| = |\pi_n(x) - r_n(x)| \geqq |\pi_n(x)| - |r_n(x)|$$
$$\geqq \frac{\mu}{2}\sigma_n - \frac{1}{4}\sigma_n \geqq \frac{\mu}{2}\sigma_n - \frac{\mu}{4}\sigma_n = \frac{\mu}{4}\sigma_n.$$

With (10) this gives

$$|L(\phi_n)| \geqq |\phi_n(x)| - \left|\int_a^b k(x,t)\phi_n(t)dt\right|$$
$$\geqq \frac{\mu}{4}\sigma_n - \mu\epsilon_n = \mu\left(\frac{\sigma_n}{4} - \epsilon_n\right),$$

or, since at least one-half of the interval $|x-x_0| \leqq (b-a)/(4n^2)$ is contained in (a, b) and since $\phi_n = r_n - \pi_n = u - P_n$,

$$\gamma_n \geqq \frac{b-a}{4n^2}\left|\mu\left(\frac{\sigma_n}{4} - \epsilon_n\right)\right|^m.$$

Under the assumption that $\epsilon_n \leqq \sigma_n/4$ it is found that

$$\sigma_n \leqq \frac{4\cdot 4^{1/m}}{(b-a)^{1/m}} n^{2/m}\gamma_n^{1/m} + 4\epsilon_n. \tag{11}$$

On the other hand, if the assumption is made that ϵ_n is greater than $\sigma_n/4$, then $\sigma_n < 4\epsilon_n$, so that (11) is generally true.

Furthermore, since $|\pi_n(x)| \leqq \mu\sigma_n$ and $|r_n(x)| \leqq \epsilon_n$, it follows that

$$|\phi_n(x)| = |r_n(x) - \pi_n(x)| \leqq \epsilon_n + \mu\sigma_n.$$

But $r_n - \pi_n$ is identical with $u - P_n$; hence

$$|u(x) - P_n(x)| \leqq \frac{4\cdot 4^{1/m}}{(b-a)^{1/m}} n^{2/m}\gamma_n^{1/m} + 4\mu\epsilon_n + \epsilon_n.$$

It is now apparent that, for n sufficiently large, since $\gamma_n \leqq (N\epsilon_n)^m$,

$$| u(x) - P_n(x) | \leqq S n^{2/m} \epsilon_n,$$

where S is a positive constant.

The previous discussion can be summed up as follows.

THEOREM 1. *If $u(x)$ is a solution of the integral equation*

$$L(u) \equiv u(x) - \int_a^b k(x, t)u(t)dt = f(x),$$

under the hypothesis on $L(u)$ already stated, and if $P_n(x)$ is the approximating polynomial to $u(x)$ determined by the least mth power method, then a sufficient condition for the convergence of $P_n(x)$ towards $u(x)$ is that it be possible to choose polynomials $q_n(x)$ for every value of n so that

$$\lim_{n \to \infty} n^{2/m} \epsilon_n = 0,$$

where ϵ_n is an upper bound for $|u(x) - q_n(x)|$ in the interval (a, b).

The last condition can be interpreted in terms of continuity of $u(x)$ and its derivatives, and these in turn will be guaranteed by imposition of suitable hypotheses on the given functions $f(x)$ and $k(x, t)$. It appears from the representation

$$u(x) = f(x) + \int_a^b k(x, t)u(t)dt$$

that if $f(x)$ has a modulus of continuity not exceeding $\omega(\delta)$, and if $k(x, t)$ as a function of x has, uniformly with respect to t, a modulus of continuity not exceeding a constant multiple of $\omega(\delta)$, then $u(x)$ likewise has a modulus of continuity not exceeding a constant multiple of $\omega(\delta)$. For the case $m > 2$, the condition that it be possible to make $n^{2/m}\epsilon_n$ approach zero will be satisfied, according to known theorems on approximation,* if $\lim_{\delta \to 0} \omega(\delta)/\delta^{2/m} = 0$. In a similar way it is seen from the representation

$$u^{(p)}(x) = f^{(p)}(x) + \int_a^b \frac{\partial^p}{\partial x^p} k(x, t)u(t)dt$$

* D. Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. XI, 1930, pp. 13–18.

for the pth derivative of $u(x)$ that, when $1<m\leqq 2$, sufficient conditions for convergence can be formulated in terms of properties of continuity of $f(x)$, $k(x, t)$, and $k_x(x, t)$, and that for $m=1$ there are corresponding conditions involving second derivatives.

4. *Convergence of the Approximating Polynomial for $m<1$.* In order to proceed with the discussion of convergence of the approximating polynomial $P_n(x)$ for $m<1$, there is occasion to add to the hypothesis already assumed for equation (1) the condition that $k_x(x, t)$, the derivative of $k(x, t)$ with respect to x, be a continuous function of the two variables in the closed region $a\leqq x\leqq b$, $a\leqq t\leqq b$.

Before presentation of the actual convergence theorem, the following auxiliary theorem will be established.

If $p_n(x)$ is an arbitrary polynomial of the nth degree, and $L(u)$ the expression defined in (1), *then, if η is the maximum of $|L(p_n)|$ on $a\leqq x\leqq b$,*

$$|p_n(x)|\leqq A\eta,$$

and

$$\left|\frac{d}{dx}L[p_n(x)]\right|\leqq Bn^2\eta$$

for all values of x in (a, b), where A and B are positive constants depending neither on n nor on the coefficients in $p_n(x)$.

Let

$$R(x)=L(p_n)=p_n(x)-\int_a^b k(x, t)p_n(t)dt.$$

Since

$$p_n(x)=R(x)+\int_a^b H(x, t)R(t)dt,$$

it follows that

$$|p_n(x)|\leqq|R(x)|+\int_a^b|H(x, t)||R(t)|dt$$

$$\leqq\left[1+\int_a^b Gdt\right]\eta=A\eta,$$

where $A=1+G(b-a)$. Moreover, as it was assumed that $k_x(x, t)$ is continuous in $a \leqq x \leqq b$, $a \leqq t \leqq b$,

$$\frac{d}{dx}[L(p_n)] = p_n'(x) - \int_a^b k_x(x, t)p_n(t)dt,$$

whence

$$\left| \frac{d}{dx}[L(p_n)] \right| \leqq |p_n'(x)| + \int_a^b |k_x(x, t)||p_n(t)|\,dt.$$

The function $p_n(x)$ is a polynomial of the nth degree; hence, by Markoff's theorem, $|p_n'(x)| \leqq N'n^2\eta$, where N' is a constant independent of η and the polynomial $p_n(x)$. It follows, therefore, if $|k_x(x, t)| \leqq M'$ in (a, b), that

$$\left| \frac{d}{dx}[L(p_n)] \right| \leqq N'n^2\eta + M'A(b-a)\eta.$$

Consequently,

$$\left| \frac{d}{dx}[L(p_n)] \right| \leqq Bn^2\eta,$$

where B is a constant.

To proceed with the convergence theorem, let x_0 be a point in $a \leqq x \leqq b$ at which $|L(\pi_n)|$ attains its maximum, where $\pi_n(x)$ has the same meaning as in the preceding section, and let this maximum be denoted by η. Then if x is interior to the interval $|x-x_0| \leqq 1/(2n^2B)$, or the part of this interval which is contained in (a, b), in case x_0 is distant from a or b by less than the amount indicated, and, if the mean value theorem is applied to $L(\pi_n)$ together with the conclusions of the above auxiliary theorem, it is seen that

$$|L[\pi_n(x)] - L[\pi_n(x_0)]| \leqq \eta/2,$$

whence $|L(\pi_n)| \geqq \eta/2$. But

$$|L(r_n)| \leqq |r_n(x)| + \int_a^b |k(x, t)||r_n(t)|\,dt \leqq |1 + M(b-a)|\,\epsilon_n.$$

So if for the time being it be assumed that $|1+M(b-a)|\,\epsilon_n \leqq \eta/4$,

$$|L(\pi_n) - L(r_n)| \geqq |L(\pi_n)| - |L(r_n)| \geqq \eta/4$$

throughout the interval specified. Hence

$$\gamma_n = \int_a^b |L(\pi_n) - L(r_n)|^m dx \geqq \frac{1}{2n^2B}\left(\frac{\eta}{4}\right)^m,$$

from which it follows that

$$\eta \leqq 4(2Bn^2\gamma_n)^{1/m}.$$

If, on the other hand, we assume that $[1+M(b-a)]\epsilon_n \geqq \eta/4$, then

$$\eta \leqq 4[1 + M(b - a)]\epsilon_n.$$

It is therefore evident that in all cases

$$\eta \leqq 4(2Bn^2\gamma_n)^{1/m} + 4[1 + M(b - a)]\epsilon_n.$$

Since by the auxiliary theorem $|\pi_n(x)| \leqq A\eta$, while $|r_n(x)| \leqq \epsilon_n$,

$$|r_n(x) - \pi_n(x)| \leqq |r_n(x)| + |\pi_n(x)| \leqq 4A(2Bn^2\gamma_n)^{1/m} + 4A[1 + M(b - a)]\epsilon_n + \epsilon_n.$$

Furthermore, since $r_n - \pi_n$ is the same as $u - P_n$ and $\gamma_n \leqq (N\epsilon_n)^m$,

$$|u(x) - P_n(x)| \leqq A'n^{2/m}\epsilon_n$$

for all values of x in the interval (a, b), where A' is a positive constant not depending on n or on ϵ_n. The following theorem can therefore be stated for the case $m < 1$.

THEOREM 2. *If in addition to the hypothesis of Theorem* 1, *the assumption is made that* $k_x(x, t)$ *is continuous in the square* $a \leqq x \leqq b$, $a \leqq t \leqq b$, *then the conclusion of Theorem* 1 *is valid for* $m < 1$.

The conditions to be imposed on $f(x)$ and $k(x, t)$, in order that it may be possible to make $n^{2/m}\epsilon_n$ approach zero, are similar to those which were mentioned after Theorem 1 for the case $1 < m \leqq 2$, with suitably modified specifications involving second derivatives or derivatives of higher order.

THE UNIVERSITY OF MINNESOTA

PLANE SECTIONS THROUGH AN ASYMPTOTIC TANGENT OF A SURFACE

BY E. P. LANE

1. *Introduction.* It is well known that the plane curve in which an analytic surface S in ordinary space is cut by a plane through an asymptotic tangent at an ordinary non-parabolic point O of S has an inflexion at O, provided that the plane is not the tangent plane of S at O. Bompiani has enriched* the projective differential geometry of a plane curve in the neighborhood of an inflexion point on it by introducing various osculants covariantly associated with the inflexion. B. Su has applied† some of Bompiani's results in connection with the geometry of curves and surfaces in ordinary space. It is the purpose of this note to supplement the investigations of Su, by studying further the loci of the osculants and various points and lines associated by Bompiani with the inflexion point of a plane curve of section of a surface made by a variable plane through a fixed asymptotic tangent of the surface at the point.

2. *Power Series Expansions.* G. M. Green calculated‡ a canonical power series expansion for one non-homogeneous projective coordinate z of a point on a surface in terms of the other two coordinates x, y. If the unit point for Green's expansion is suitably transformed, his expansion takes the form

$$
\begin{aligned}
z = {} & xy + x^3 + \cdots + Bx^3y + \cdots + x^5 + Cx^4y \\
& + \cdots + Ex^6 + Fx^5y + \cdots + Gx^7 \\
& + \cdots + Ix^8 + \cdots ,
\end{aligned} \tag{1}
$$

the omitted terms being immaterial for our purposes. The coefficients are absolute invariants of the surface. The line $x=y=0$ is *the first canonical edge* of Green, while the opposite edge of

* Bompiani, *Per lo studio proiettivo-differenziale delle singularità*, Bollettino dell'Unione Matematica Italiana, vol. 5 (1926), p. 118.

† B. Su, *On certain quadratic cones projectively connected with a space curve and a surface*, Tôhoku Mathematical Journal, vol. 38 (1933), p. 233.

‡ Green, *Memoir on the general theory of surfaces and rectilinear congruences*, Transactions of this Society, vol. 20 (1919), p. 79.

the tetrahedron of reference is Green's *second canonical edge*, at the origin O (0, 0, 0). Let homogeneous coordinates be introduced by the definitions

$$x = \frac{x_2}{x_1}, \qquad y = \frac{x_3}{x_1}, \qquad z = \frac{x_4}{x_1}.$$

Then the vertex (0, 0, 0, 1) of the tetrahedron is the point, distinct from O, in which the first canonical edge meets the canonical quadric of Wilczynski.

The equation

$$y = nz, \qquad (n \neq 0), \tag{2}$$

represents a plane through an asymptotic tangent, $y=z=0$, at the origin O. The plane (2) cuts the surface (1) in a curve whose projection, C, from the vertex (0, 0, 0, 1) onto the tangent plane, $z=0$, is represented by the equation obtained by eliminating z between equations (1), (2). If this equation is solved for y as a power series in x the result is

$$y = n[x^3 + nx^4 + (n^2 + 1)x^5 + b_6x^6 + b_7x^7 + b_8x^8 + \cdots], \tag{3}$$

in which the coefficients $b_6, \ldots, b_8$ are defined by the formulas

$$\begin{aligned} b_6 &= n^3 + (B + 1)n + E, \\ b_7 &= n^4 + (2B + 1)n^2 + (C + E)n + G, \\ b_8 &= n^5 + (3B + 1)n^3 + (2C + E)n^2 + (B + F + G)n + I. \end{aligned} \tag{4}$$

The origin is obviously an inflexion on the curve C, and hence is also an inflexion on the curve of section, as was remarked at the outset.

3. *Bompiani's Osculants at an Inflexion.* The power series (3) can be employed to calculate the equations of the osculants and other lines and points associated by Bompiani with the inflexion O of the curve C. First of all, the equation of *the seven-point cusped cubic* is found to be

$$l^2 \frac{y}{n} - l^3 = 0, \tag{5}$$

where l, t are two linear functions defined by

$$(6)\qquad \begin{aligned} l &= x + \frac{1}{12}(n^2+4)\frac{y}{n}, \\ t &= 1 - \frac{1}{2}nx - \frac{1}{2}[(B-1)n + E]\frac{y}{n}. \end{aligned}$$

The equation of *the cusp-locus*, that is, the locus of the cusps of all six-point cusped cubics, is $l=0$. The equation of *the cusp-tangent* of the seven-point cusped cubic (5) is $t=0$. The coordinates of *the cusp* O_2 of the seven-point cusped cubic are found, by solving simultaneously the equations $l=0$, $t=0$, to be

$$(7)\qquad \left(\frac{2(n^2+4)}{n^3 - 4(3B-4)n - 12E}, \quad \frac{-24n}{n^3 - 4(3B-4)n - 12E}\right).$$

Finally, the coordinates of *the point* O_1 in which the cusp-tangent, $t=0$, intersects the inflexional tangent, $y=0$, of the curve C are found to be

$$(8)\qquad \left(\frac{2}{n}, \quad 0\right).$$

The equation of *the eight-point cubic through the cusp* O_2 is

$$(9)\qquad t^2\frac{y}{n} - l^3 - Ul\frac{y^2}{n^2} = 0,$$

where U is a polynomial defined by

$$(10)\qquad \begin{aligned} U = &-\frac{1}{48}n^4 - \frac{1}{6}(3B-2)n^2 + \frac{1}{2}(2C-3E)n \\ &+ \frac{1}{3}(3G-4). \end{aligned}$$

The equation of *the nine-point quartic curve which passes through the point* O_1 *tangent to the cusp-tangent,* $t=0$, *and which has a node at the point* O_2 *with the cusp-locus,* $l=0$, *for one nodal tangent* is

$$(11)\qquad t\left(t^2\frac{y}{n} - l^3 - Ul\frac{y^2}{n^2}\right) - Vl^2\frac{y^2}{n^2} = 0,$$

where V is a polynomial defined by

$$V = -(C - E)n^2 - (2B - F + 2G - 3)n - 3E + I. \tag{12}$$

The equation of *the residual nodal tangent* of this quartic is

$$Ut + Vl = 0. \tag{13}$$

Finally, the equation of *the line joining the origin O to the intersection, distinct from O and O_2, of the quartic* (11) *and the seven-point cusped cubic* (5) is

$$V^2 \frac{y}{n} - U^2 l = 0. \tag{14}$$

Direct substitution from equation (3) into the left members of equations (5) and (9) leads to

$$\begin{aligned} t^2 \frac{y}{n} - l^3 &= Ux^7 + \cdots, \\ t^2 \frac{y}{n} - l^3 - Ul\frac{y^2}{n^2} &= Vx^8 + \cdots, \end{aligned} \tag{15}$$

the omitted terms being of higher order than those written. It follows from the first of these equations that *the seven-point cusped cubic* (5) *has eight-point contact with the curve C at the point O in case* $U=0$. Equation (10) shows that there are four values of n satisfying the equation $U=0$. Therefore *through each asymptotic tangent at a point of a surface there are four planes producing sections for which the seven-point cusped cubic has eight-point contact.* For such a section the cubics (5), (9) are the same, and the quartic (11) has a cusp with the line $l=0$ for cusp tangent.

It follows from the second of equations (15) that *the eight-point cubic* (9) *has nine-point contact with the curve C at the point O in case* $V=0$. Equation (12) shows that there are two values of n satisfying the equation $V=0$. Therefore *through each asymptotic tangent at a point of a surface there are two planes producing sections for which the eight-point cubic through the associated point O_2 has nine-point contact.* In this case the quartic (11) is composite.

4. *Loci of Bompiani's Osculants.* The loci of Bompiani's osculants and associated lines and points for a plane section made by a variable plane through a fixed asymptotic tangent will now be considered. First of all, parametric equations for the locus of the cusp of the seven-point cusped cubic may be written by setting x and y respectively equal to the coordinates given in (7) and adjoining the equation $z=y/n$. Thus the locus is seen to be a twisted cubic curve. In a similar way the point O_1 is seen to generate on the tangent $y=z=0$ a point-row which is projective with the pencil of planes generated by the plane (2) and having this tangent for axis. Finally, the locus of the intersection, distinct from O and O_2, of the quartic and the seven-point cubic, is a unicursal curve of order fifteen.

The equation of the locus of *the cusp-locus* is found by eliminating n between equation (2) and the equation $l=0$, with l defined by the first of (6). The result is $12zx+y^2+4z^2=0$. Therefore this locus is a quadric cone, as Su has shown. This cone is tangent to the tangent plane, $z=0$, along the asymptotic tangent $y=z=0$.

For the purpose of economy in writing, let us define some polynomials p_2, q_2, r_2, p_4, p_6, q_6, r_6, p_8, p_{10} by the formulas

$$
\begin{aligned}
p_2 &= zx + \frac{1}{12}y^2 + \frac{1}{3}z^2,\\
q_2 &= z - \frac{1}{2}xy - \frac{1}{2}[(B-1)yz + Ez^2],\\
r_2 &= -(C-E)y^2 - (2B - F + 2G - 3)yz - (3E - I)z^2,\\
p_4 &= -\frac{1}{48}y^4 - \frac{1}{6}(3B-2)y^2z^2 + \frac{1}{2}(2C - 3E)yz^3\\
&\qquad + \frac{1}{3}(3G-4)z^4,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
p_6 &= p_4q_2 + z^2r_2p_2,\\
q_6 &= z^2q_2^2 - p_2^3,\\
r_6 &= z^2q_2^3 - p_2^3 - p_4p_2,\\
p_8 &= q_2r_6 - z^2r_2p_2^2,\\
p_{10} &= z^6r_2^2 - p_4^2p_2.
\end{aligned}
$$

Then the equation of the locus of the cusp-locus is $p_2=0$, as was just shown. Moreover, the equation of the locus of *the cusp-tangent* is $q_2=0$, so that this locus is a quadric surface intersecting the tangent plane, $z=0$, in the asymptotic tangents, $z=xy=0$. The plane (2) is tangent to this quadric at the point $O_1(2/n, 0)$; therefore the projective correspondence mentioned above in discussing the locus of the point O_1 is the correlation between points on the tangent $y=z=0$ and the tangent planes of the quadric $q_2=0$ at these points. The equation of the locus of *the residual nodal tangent* of the nine-point quartic is $p_6=0$, so that this locus is an algebraic ruled surface of the sixth order, having the tangent $y=z=0$ for quintuple line and containing the other asymptotic tangent, $z=x=0$. The equation of the locus of *the line joining the point O to the other intersection of the nine-point quartic and the seven-point cusped cubic* is $p_{10}=0$, so that this locus is an algebraic cone of the tenth order, with the tangent $y=z=0$ as ten-fold line.

The equation of the locus of *the seven-point cusped cubic* is $q_6=0$, so that this locus is an algebraic surface of the sixth order. The equation of the locus of *the eight-point cubic* is $r_6=0$, so that this locus is an algebraic surface of the sixth order. Finally, the equation of the locus of *the nine-point quartic* is $p_8=0$, so that this locus is an algebraic surface of the eighth order.

Many further properties and relations of the loci here defined can be read off immediately from their equations, but space does not permit their inclusion here.

UNIVERSITY OF CHICAGO

ON THE INDEPENDENCE OF UNDEFINED IDEAS

BY J. C. C. MCKINSEY

Considerable attention has been devoted recently to the independence of postulates, but little notice has been paid to the question of the independence of the undefined ideas occurring in postulate sets. It is true that Padoa* has given a definition of independence of undefined ideas and has constructed a test to show its presence; but, during the thirty-four years which have followed his work, no one, so far as I know, has discussed the concept or applied the test. I therefore feel that it may not be out of place again to define this concept and to attempt to justify the test for it. I also undertake to exhibit the importance of the independence of undefined ideas by showing the situation which arises when it is lacking.

Suppose S is an abstract mathematical system with undefined ideas $(K_1, K_2, \cdots ; R_1, R_2, \cdots ; \odot_1, \odot_2, \cdots)$, where K_i represents an undefined class, R_i an undefined relation, and $\odot_i$ an undefined operation. Then we say that any one of the classes, as K_1, is dependent on the other undefined ideas if there exists in S a theorem such as the following:

$$(1) \quad (x \in K_1) \equiv F(x; K_2, \cdots ; R_1, R_2, \cdots ; \odot_1, \odot_2, \cdots),$$

where the triple bar indicates mutual implication. Similarly we say that R_1 is dependent if there exists in S a theorem such as

$$(2) \quad (xR_1y) \equiv G(x, y; K_1, K_2, \cdots ; R_2, \cdots ; \odot_1, \odot_2, \cdots),$$

and that $\odot_1$ is dependent if there exists in S a theorem such as

$$(3) \quad (x \odot_1 y) = H(x, y; K_1, K_2, \cdots ; R_1, R_2, \cdots ; \odot_2, \cdots),$$

where the double bar indicates identity. (It is important to notice that, in (1), the only undefined symbol occurring on the left is K_1, and that K_1 does not occur on the right; similarly for R_1 and $\odot_1$, respectively, in (2) and (3)). If an undefined idea is not dependent, it is said to be independent.

* *Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque*, Bibliothéque du Congrès International de Philosophie, vol. 3 (1900).

The importance of the independence of undefined ideas lies in the fact that if notions are not independent it is possible to define one of them in terms of the others without changing the character of S. And such a definition will, in general, make it possible to reduce the number of postulates required for S. This is true because, so far as deductive power is concerned, it makes no difference whether a proposition is given as a postulate or as a definition.

The following example will perhaps make this clearer: Huntington* has given a complete set of postulates for the algebra of complex quantities, in which he takes as undefined the five ideas $(K, C, \oplus, \odot, \leqslant)$, where K is the class of complex quantities, and C the sub-class of real quantities. From the postulates given it is possible to deduce the following theorem.

A necessary and sufficient condition that $a \leqslant b$ is that the following three propositions be true:

1. *a and b belong to C,*
2. *$a \neq b$,*
3. *there exists a t, belonging to C, such that $a \oplus (t \odot t) = b$.*

Since this theorem holds in the system, it is evident that we can *define* the relation $\leqslant$ as follows:

$$a \leqslant b :=: : (a, b \in C) : : (a \neq b) : : (\exists t) : (t \in C) \cdot (a \oplus (t \odot t) = b). \quad \textit{Def.}$$

For this definition would not allow the deduction of anything which would not follow, anyhow, from the theorem. Also, although Huntington's postulates, without the definition, are independent, they become dependent in the presence of the definition. For P15 (which asserts that if $a \leqslant b$, then a and b are distinct) is given directly in the definition. And it is easy to deduce the following postulates from other postulates in the presence of the definition.

P18. *If $x \leqslant y$, then $a \oplus x \leqslant a \oplus y$.*

P19. *If $z \leqslant a$ and $z \leqslant b$, where z is the zero element, then $z \leqslant a \odot b$.*

If the ideas are independent, on the other hand, then to define one of them in terms of the others is to change the character of S from its character as given by the postulates. This does not mean, it should be noticed, that it is not possible to give a

* *Monographs on Topics of Modern Mathematics*, pp. 186–190, or Transactions of this Society, vol. 6 (1905), pp. 209–229.

definition in such a case, but merely that it is impossible to give a definition without introducing new information. Thus suppose, for example, that our system* involves only the two undefined ideas K and $\oplus$ and the single postulate: *K consists of just two elements*, 0 *and* 1. Then it is seen that $\oplus$ is independent of K in the sense defined above, but clearly we may define $\oplus$ in any one of sixteen ways by means of a table

$\oplus$	0	1
0	l_1	l_2
1	l_3	l_4

, $(l_i = 0 \text{ or } 1)$.

But it is evident that making any one of the possible definitions will change the nature of the system, since any definition will enable us to prove propositions which could not be proved from the postulate alone. If the primitive ideas of a categorical system are independent, however, then any definition of one of them in terms of the others will make the system inconsistent.

I now set forth a method whereby, if K_1 is independent of $(K_2, \cdots ; R_1, R_2, \cdots ; \odot_1, \odot_2, \cdots)$, this independence may be shown. Let $\overline{S}$ be a concrete representation of S, so that K_i is interpreted concretely as $\overline{K}_i$, R_i as $\overline{R}_i$, and $\odot_i$ as $\overline{\odot}_i$. Let $\overline{\overline{S}}$ be a second concrete interpretation of S, so that K_1 is interpreted as $\overline{\overline{K}}_1$, K_i as $\overline{K}_i$ for $i \neq 1$, R_i as $\overline{R}_i$, and $\odot_i$ as $\overline{\odot}_i$. Let $\overline{K}_i$ and $\overline{\overline{K}}_1$ differ in their *extension*, so that there is an element a which belongs to $\overline{K}_1$ but not to $\overline{\overline{K}}_1$. If it is possible to find $\overline{S}$ and $\overline{\overline{S}}$ as specified, then K_1 is independent of $(K_2, \cdots ; R_1, R_2, \cdots ; \odot_1, \odot_2, \cdots)$. For if K_1 were dependent, then we should have

(4) $\quad (x \epsilon K_1) \equiv F(x; K_2, \cdots ; R_1, R_2, \cdots ; \odot_1, \odot_2, \cdots)$.

Hence, in $\overline{S}$,

(5) $\quad (x \epsilon \overline{K}_1) \equiv F(x; \overline{K}_2, \cdots ; \overline{R}_1, \overline{R}_2, \cdots ; \overline{\odot}_1, \overline{\odot}_2, \cdots)$.

And in $\overline{\overline{S}}$,

(6) $\quad (x \epsilon \overline{\overline{K}}_1) \equiv F(x; \overline{K}_2, \cdots ; \overline{R}_1, \overline{R}_2, \cdots ; \overline{\odot}_1, \overline{\odot}_2, \cdots)$.

Thus, since F is single-valued, and since material equivalence is transitive,

* I am indebted to B. A. Bernstein for this example, and for several other suggestions.

$$x \epsilon \overline{K}_1 \equiv x \epsilon \overline{\overline{K}}_1. \tag{7}$$

Hence, in particular,

$$a \epsilon \overline{K}_1 \equiv a \epsilon \overline{\overline{K}}_1. \tag{8}$$

But by hypothesis $a \epsilon K_1$ is true and $a \epsilon \overline{\overline{K}}_1$ is false, so that (8) is absurd. Hence, if the systems $\overline{S}$ and $\overline{\overline{S}}$ can be found, K_1 is independent.

In a similar way, if we can find two interpretations $(\overline{K}_1, \overline{K}_2, \cdots ; \overline{R}_1, \overline{R}_2, \cdots ; \overline{\odot}_1, \overline{\odot}_2, \cdots)$ and $(\overline{K}_1, \overline{K}_2, \cdots ; \overline{\overline{R}}_1, \overline{\overline{R}}_2, \cdots ; \overline{\overline{\odot}}_1, \overline{\overline{\odot}}_2, \cdots)$, then an argument like that used above will show that R_1 is independent; and similarly for the operations. It is important to notice, however, that in order that $\overline{R}_1$ be different from $\overline{\overline{R}}_1$, it must differ in extension, that is, there must be an ordered couple (a, b) such that $a \overline{R}_1 b$ is true and $a \overline{\overline{R}}_1 b$ is false. And in order that $\overline{\odot}_1$, differ from $\overline{\overline{\odot}}_1$ there must be an ordered triple (a, b, c) such that*

$$a \overline{\odot}_1 b = c, \qquad \text{and} \qquad a \overline{\overline{\odot}}_1 b \neq c.$$

I now illustrate the application of the method by an example. For this purpose I make use of the following system S, due to Huntington.†

Undefined ideas:

K = *an undefined class of elements*, $a, b, c, \cdots$.

C = *an undefined subclass within* K.

$a+b$ = *the result of an undefined binary operation on* a *and* b.

a' = *the result of an undefined unary operation on* a.

It is assumed that C, and hence K, is non-empty.

POSTULATES.

P1. *If* a *is in* K *and* b *is in* K, *then* $a+b$ *is in* K.

P2. *If* a *is in* K, *then* a' *is in* K.

P3. *If* a *is in* C, *then* a *is in* K.

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* Padoa's failure to indicate any criterion for the "differentness" of two interpretations is perhaps more than anything else responsible for the reluctance of mathematicians to employ his test.

† *Independent postulates for the informal part of Principia Mathematica*, this Bulletin, vol. 40 (1934), pp. 127–136.

P4. *If $a+b$ is in C, then $b+a$ is in C.*
P5. *If a is in C, then $a+b$ is in C.*

DEFINITION. *The notation (a is in C') shall mean (a is in K and a is not in C).*

P6. *If a is in K and a' is in C, then a is in C'.*
P7. *If a is in K and a' is in C', then a is in C.*
P8. *If $a+b$ is in C and a' is in C, then b is in C.*

I now show that the four ideas ($+$, $'$, C, K) are independent.

INDEPENDENCE OF $+$

$S_{1.1}$ $K = 1, 2, 3$; $C = 1, 2$

$+$	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

a	a'
1	3
2	3
3	1

$S_{1.2}$ $K = 1, 2, 3$; $C = 1, 2$

$+$	1	2	3
1	1	1	1
2	1	1	1
3	1	1	3

a	a'
1	3
2	3
3	1

Thus it is seen that there are two systems $S_{1.1}$ and $S_{1.2}$, each of which satisfies P1-P8 and in each of which K, C, and $'$ have the same specification, while $+$ is specified differently in $S_{1.1}$ and $S_{1.2}$. Hence $+$ cannot be defined in terms of K, C, and $'$ without changing the character of S.

INDEPENDENCE OF $'$

$S_{2.1}$ $K = 1, 2, 3$; $C = 1, 2$

$+$	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

a	a'
1	3
2	3
3	1

$S_{2.2}$ $K = 1, 2, 3$; $C = 1, 2$

$+$	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

a	a'
1	3
2	3
3	2

INDEPENDENCE* OF C

$S_{3.1}$: $K = 1, 2, 3, 4$; $C = 1, 2$

+	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
4	1	2	3	4

a	a'
1	4
2	3
3	2
4	1

$S_{3.2}$: $K = 1, 2, 3, 4$; $C = 1, 3$

+	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
4	1	2	3	4

a	a'
1	4
2	3
3	2
4	1

INDEPENDENCE OF K

$S_{4.1}$: $K = 1, 2$; $C = 1$

+	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

a	a'
1	2
2	1
3	1

$S_{4.2}$: $K = 1, 2, 3$; $C = 1$

+	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

a	a'
1	2
2	1
3	1

Thus it follows that the four undefined ideas are all independent.

I may remark, in conclusion, that it is possible to construct a theory of the "irredundancy of undefined ideas" which is analogous to Church's "irredundancy of postulates." Undefined classes are irredundant when no one of them is a sub-class of some of the others, and similarly for relations and operations (taken in extension). Thus, in either of the examples above, C is not irredundant, since it is a sub-class of K; though this

* $S_{3.1}$ is given by Huntington, loc. cit., to show the consistency of P1-P8.

situation could easily be avoided by taking C and C' as undefined and defining K as $C+C'$. It is doubtful, however, whether irredundancy of undefined ideas is an especially useful concept.

THE UNIVERSITY OF CALIFORNIA

DISTRIBUTIONS OF GREATEST VARIATES, LEAST VARIATES, AND INTERVALS OF VARIATION IN SAMPLES FROM A RECTANGULAR UNIVERSE*

BY E. G. OLDS

1. *Introduction.* It is proposed to present in this paper the distributions of greatest variates, least variates, and intervals of variation, in samples of size N drawn, without replacement, from the population characterized by the frequency distribution

$$f(x) = \begin{cases} 1 \text{ for } x = 0, 1, 2, \cdots, b, \\ 0 \text{ elsewhere.} \end{cases}$$

This is a finite universe of discrete variates, distributed rectangularly.

The distributions of various statistical parameters, in the case of samples from rectangular distributions, have been investigated by Rietz† and others,‡ but they have been concerned with continuous distributions. The two investigations most closely related to the contents of this paper are those of J. Neyman§ and E. S. Pearson, and of P. R. Rider.¶

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* Presented to the Society, December 27, 1934.

† *On a certain law of probability of Laplace*, Proceedings of the International Mathematical Congress, Toronto (1924), vol. 2, pp. 795–799.

‡ Philip Hall, *The distribution of means for samples of size N drawn from a population in which the variate takes values between* 0 *and* 1, *all such values being equally probable*, Biometrika, vol. 19 (1927), pp. 240–244. Allen T. Craig, *On the distributions of certain statistics*, American Journal of Mathematics, vol. 54 (1932), pp. 353–366.

§ *On the use and interpretation of certain test criteria for purposes of statistical inference*, Biometrika, vol. 20A (1928), pp. 175–240.

¶ *On the distribution of the ratio of the mean to standard deviation in small samples from non-normal universes*, Biometrika, vol. 21 (1929), pp. 124–143.

Neyman* and Pearson consider samples of n from a continuous, rectangular distribution of range w. Using W to represent the range in the sample, they find the distribution of W,

$$\phi(W) = n(n-1)\frac{W^{n-2}}{w^{n-1}}\left(1-\frac{W}{w}\right). \tag{1}$$

Rider† assumes an infinite rectangular population of discrete variates and considers the distribution of range in samples of four from a ten-class universe. He obtains the distribution,

$$p(W) = 0.001(-12W^3 + 120W^2 - 2W + 20), \tag{2}$$

and compares it with the distribution of Neyman and Pearson, evaluating (1) for $n=4$ and $w=10$. He notes that the distribution of range and other statistical parameters, for the discrete universe, seem to differ little from those obtained previously for the corresponding continuous universe. It will be interesting to note whether this statement needs to be qualified in the case of the finite universe under consideration.

2. *Greatest Variate.* In a sample of size N, consider, first, the probability, $p(N-1)$, for a greatest variate of size $N-1$. Since the sample must contain the integers from 0 to $N-1$, both inclusive,

$$p(N-1) = \frac{1}{C_{b+1,N}}. \tag{3}$$

If any of the N numbers in this sample be replaced by the number N, the resulting sequence will have the number N as its greatest variate. Corresponding to each sample having $N-1$ as the greatest variate are samples having the number N as greatest variate. This provides a convenient way of enumerating the possible samples which have N as a greatest variate, and we can write

$$p(N) = N \cdot p(N-1). \tag{4}$$

Likewise, the samples with $N+1$ as the greatest variate may be exhibited by replacing each of the numbers in the first-mentioned

* Loc. cit., p. 210.
† Loc. cit., pp. 136–137.

sample, in succession, by the number $N+1$ and choosing the rest of the sample from the numbers less than $N+1$. Continuing this process, it is simple to get the remaining samples, including the cases where the greatest variate is b. We have

$$p(N+1) = C_{N+1,2}\, p(N-1), \tag{5}$$

and, in general,

$$p(N+k) = C_{N+k,k+1}\, p(N-1), \qquad (k = -1, 0, \cdots, b-N). \tag{6}$$

Replacing $p(N-1)$ in (6) by its value from (3), and replacing $N+k$ by G, we find that the required distribution is

$$p(G) = \frac{C_{G,\,G-N+1}}{C_{b+1,\,N}}, \tag{7}$$

where G may assume all integral values from $N-1$ to b. This may be written more compactly as

$$p(G) = \frac{N(G)^{(N-1)}}{(b+1)^{(N)}}. \tag{8}$$

The moments of this distribution about the origin may be computed by summing $G^n p(G)$ over the possible values of G, and, by ordinary methods, the moments about the mean may then be obtained. We find

$$M_G = \frac{N(b+1)}{N+1} - \frac{1}{N+1}, \tag{9}$$

and the variance,

$$\sigma_G^2 = \frac{N(b+1)^2}{(N+2)(N+1)^2} - \frac{(N^2-N)b + (2N^2-N)}{(N+2)(N+1)^2}. \tag{10}$$

It is interesting to compare these results with those obtained from a continuous distribution. Consider the distribution with the probability function

$$\phi(x) = \frac{1}{b+1}, \qquad (0 \leqq x \leqq b+1).$$

Then the probability that the greatest variate in a sample is between G and $G+dG$ is K times the product of the probabilities of $N-1$ members being less than G and one variate lying between G and $G+dG$. Thus*

$$\Phi(G)dG = K\cdot\left[\int_0^G \frac{dx}{b+1}\right]^{N-1}\cdot\frac{dG}{b+1} = \frac{NG^{N-1}}{(b+1)^N}\,dG, \tag{11}$$

since

$$\int_0^{b+1}\Phi(G)dG = 1.$$

Also

$$\int_0^{b+1} G\Phi(G)dG = \frac{N(b+1)}{N+1}, \tag{12}$$

and

$$\int_0^{b+1}\left[G - \frac{N(b+1)}{N+1}\right]^2\Phi(G)dG = \frac{N(b+1)^2}{(N+2)(N+1)^2}. \tag{13}$$

Compare these three results with (8), (9), (10) above.

3. *Least Variate.* This distribution is found by the same method as used for greatest variate. When the sample consists of the N greatest variates in the population, the smallest is $b-N+1$. It is then apparent that the probability, $r(b-N+1)$, of this least variate is given by

$$r(b-N+1) = \frac{1}{C_{b+1,N}}. \tag{14}$$

Then, reasoning as before,

$$r(b-N) = N\cdot r(b-N+1), \tag{15}$$

and, finally,

$$\begin{aligned} r(b-N-k) &= C_{N+k,k+1}\cdot r(b-N+1) \\ &= \frac{N(N+k)^{(N-1)}}{(b+1)^{(N)}}. \end{aligned} \tag{16}$$

* See E. L. Dodd, *Functions of measurement under a general law of error*, Skandinavisk Aktuarietidskrift, 1922.

If L represents the least variate, then, replacing $b-N-k$ by L,

$$r(L) = \frac{N(b-L)^{(N-1)}}{(b+1)^{(N)}}. \tag{17}$$

Proceeding as before, we obtain the mean,

$$M_L = \frac{b-N+1}{N+1}, \tag{18}$$

and the second moment about the mean,

$$\sigma_L^2 = \frac{N(b+1)^2}{(N+2)(N+1)^2} - \frac{(N^2-N)b+(2N^2-N)}{(N+2)(N+1)^2}. \tag{19}$$

It is not surprising that $M_G+M_L=b$ and $\sigma_G^2=\sigma_L^2$.

4. *Interval of Variation.* The least interval of variation, or range, occurs whenever the N numbers in the sample are consecutive. The least number in the sample may be any of the integers $0, 1, 2, \cdots, b-N+1$. Then the probability, $q(N-1)$, of a range $N-1$, is given by

$$q(N-1) = \frac{1}{C_{b+1,N}}(b-N+2). \tag{20}$$

Ranges of size N occur when the largest and smallest variates in the sample differ by N. If the smallest variate is a, the largest is $a+N$, and between the two must lie $N-2$ of the $N-1$ numbers which lie between a and $a+N$. So connected with a and $a+N$ are $N-1$ samples. Furthermore, a may be chosen in $(b-N+1)$ ways, since the greatest admissible value of $a+N$ is b. Therefore

$$q(N) = \frac{(b-N+1)C_{N-1,N-2}}{C_{b+1,N}} = \frac{(b-N+1)C_{N-1,1}}{C_{b+1,N}}. \tag{21}$$

Pursuing the same line of reasoning, we write the general term,

$$q(N+k) = \frac{(b-N-k+1)\,C_{N+k-1,k+1}}{C_{b+1,N}}, \tag{22}$$

which describes the distribution if k varies from -1 to $b-N$. If $N+k$ is replaced by R,

$$q(R) = \frac{(b - R + 1)\, C_{R-1,\, R-N+1}}{C_{b+1,\, N}}, \tag{23}$$

and, after obvious simplification,

$$q(R) = \frac{N(N-1)}{(b+1)^{(N)}} \left[(b+1)(R-1)^{(N-2)} - R^{(N-1)}\right], \tag{24}$$

where R may assume integral values from $N-1$ to b. Proceeding as before, we calculate

$$M_R = \frac{(N-1)(b+1)}{N+1} + \frac{N-1}{N+1}, \tag{25}$$

and

$$\sigma_R^2 = \frac{2(N-1)(b+1)^2}{(N+1)^2(N+2)} - \frac{2(N-1)\left[(N-1)(b+1)+N\right]}{(N+1)^2(N+2)}. \tag{26}$$

If in the formulas derived by Neyman and Pearson (see (1) and reference cited), w is replaced by $b+1$, W by R, and n by N, they become

$$\begin{aligned} \phi(R) &= \frac{N(N-1)R^{N-2}}{(b+1)^{N-1}}\left(1 - \frac{R}{b+1}\right) \\ &= \frac{N(N-1)}{(b+1)^N}\left[(b+1)R^{N-2} - R^{N-1}\right], \end{aligned} \tag{27}$$

$$\text{mean} = \frac{(N-1)(b+1)}{N+1}, \tag{28}$$

$$\text{variance} = \frac{2(N-1)(b+1)^2}{(N+1)^2(N+2)}, \tag{29}$$

and the similarity to (24), (25), (26) becomes more apparent.

Also, replacing W by R in (2) and setting N equal to four and b equal to nine in (24) and (27), we have

$$\phi(R) = \frac{1}{1000}(-12R^3 + 120R^2),$$

(for continuous distribution),

$$p(R) = \frac{1}{1000}(-12R^3 + 120R^2 - 2R + 20), \tag{30}$$

(for discrete variates and infinite classes),

$$q(R) = \frac{1}{420}(-R^3 + 13R^2 - 32R + 20),$$

(for discrete variates and unit classes).

The third distribution is quite different from the first two. This fact becomes more apparent upon examination of Table I below. If the distribution is of the type assumed in this paper, the true value for the mean range of samples of four is 6.6. Rider's distribution has a mean of 5.93, while that of Neyman-Pearson has a mean of 6. Therefore, either of the latter, if accepted as an estimate for the true mean, gives a result somewhat too small.

5. *Application.* This work was suggested by the problem of sampling automobile license numbers, in states where letters are not combined with the numbers. It was desired to estimate total registration by means of a small sample. If the greatest variate, G_s, observed in a sample, is assumed to be equal to the mean of the greatest variates from all samples, we have

$$G_s = M_G = \frac{N(b+1)}{N+1} - \frac{1}{N+1}. \tag{31}$$

Solving for b, we obtain as one estimate

$$b = \frac{(N+1)(G_s+1)}{N} - 2. \tag{32}$$

TABLE I.* Distribution of Ranges of Samples of Four from a Rectangular Universe.

Range R	$\int\phi(R)dR$	$p(R)$	$q(R)$
0	.0005	.0010	—
1	.0115	.0126	—
2	.0388	.0400	—
3	.0757	.0770	.0333
4	.1150	.1164	.0857
5	.1495	.1510	.1429
6	.1720	.1736	.1905
7	.1753	.1770	.2143
8	.1522	.1540	.2000
9	.0955	.0974	.1333
10	.0140	—	—

$\int\phi(R)dR$ = probability, for a continuous universe, that range will fall in the given class interval.

$p(R)$ = probability of given range for discrete universe of 10 classes, each class having an infinite number of variates.

$q(R)$ = probability of given range for discrete universe of 10 classes, each class having a single variate.

Similarly, the range in a sample, R_s, may be used for a second estimate, giving

$$b = \frac{N+1}{N-1} R_s - 2. \tag{33}$$

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* The first part of this table was given by Rider, (Table XII, loc. cit., p. 136). The last column has been added for the purpose of comparison.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

STANFORD UNIVERSITY, April 6, 1935.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than March 9. All these will appear in the March issue of this Bulletin.

NEW YORK CITY, April 19-20, 1935.

Abstracts must be in the hands of Associate Secretary J. R. Kline, Low Library, Columbia University, New York City, not later than March 23. Abstracts received by the Secretaries on or before March 9 will appear in the March issue of this Bulletin. There will be a symposium on the topic *The problem of three bodies, and allied problems*, at which the speakers will be Professors G. D. Birkhoff, E. W. Brown, and Aurel Wintner.

CHICAGO, ILLINOIS, April 19-20, 1935.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 23. Abstracts received by the Secretaries on or before March 9 will appear in the March issue of this Bulletin. By invitation of the Program Committee, Professor L. M. Graves will deliver an address on *Topics in Functional Calculus*.

ANN ARBOR, MICHIGAN, SUMMER MEETING AND COLLOQUIUM, September 10-14, 1935.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than August 6. Abstracts received by July 6 will appear in the July issue of the Bulletin. Professor H. S. Vandiver will deliver a series of colloquium lectures on *Fermat's last theorem and related topics in number theory*. By invitation of the Program Committee, Professor G. Y. Rainich will deliver an address entitled *Product integrals*, and Professor G. T. Whyburn an address entitled *On the structure of continua*.

R. G. D. RICHARDSON, *Secretary of the Society*.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1934, p. 66), are payable to the Treasurer of the Society, Professor G. W. Mullins, Low Library, Columbia University, New York City.

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